

# WEYL GROUPS OF FINE GRADINGS ON SIMPLE LIE ALGEBRAS OF TYPES $A$ , $B$ , $C$ AND $D$

ALBERTO ELDUQUE\* AND MIKHAIL KOCHETOV\*\*

**ABSTRACT.** Given a grading  $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$  on a nonassociative algebra  $\mathcal{L}$  by an abelian group  $G$ , we have two subgroups of  $\text{Aut}(\mathcal{L})$ : the automorphisms that stabilize each component  $\mathcal{L}_g$  (as a subspace) and the automorphisms that permute the components. By the Weyl group of  $\Gamma$  we mean the quotient of the latter subgroup by the former. In the case of a Cartan decomposition of a semisimple complex Lie algebra, this is the automorphism group of the root system, i.e., the so-called extended Weyl group. A grading is called fine if it cannot be refined. We compute the Weyl groups of all fine gradings on simple Lie algebras of types  $A$ ,  $B$ ,  $C$  and  $D$  (except  $D_4$ ) over an algebraically closed field of characteristic different from 2.

## 1. INTRODUCTION

In [EKb], we computed the Weyl groups of all fine gradings on matrix algebras, the Cayley algebra  $\mathbb{C}$  and the Albert algebra  $\mathcal{A}$  over an algebraically closed field  $\mathbb{F}$  ( $\text{char } \mathbb{F} \neq 2$  in the case of the Albert algebra). It is well known that  $\text{Der}(\mathbb{C})$  is a simple Lie algebra of type  $G_2$  ( $\text{char } \mathbb{F} \neq 2, 3$ ) and  $\text{Der}(\mathcal{A})$  is a simple Lie algebra of type  $F_4$  ( $\text{char } \mathbb{F} \neq 2$ ). Since the automorphism group schemes of  $\mathbb{C}$  and  $\text{Der}(\mathbb{C})$ , respectively  $\mathcal{A}$  and  $\text{Der}(\mathcal{A})$ , are isomorphic, the classification of fine gradings on  $\text{Der}(\mathbb{C})$ , respectively  $\text{Der}(\mathcal{A})$ , is the same as that on  $\mathbb{C}$ , respectively  $\mathcal{A}$  [EKa] and, moreover, the Weyl groups of the corresponding fine gradings are isomorphic. The situation with fine gradings on the simple Lie algebras belonging to series  $A$ ,  $B$ ,  $C$  and  $D$  is more complicated, because the fine gradings on matrix algebras yield only a part of the fine gradings on the simple Lie algebras of series  $A$  (so-called Type I gradings). In order to obtain the fine gradings for series  $B$ ,  $C$  and  $D$  and the remaining (Type II) fine gradings for series  $A$ , one has to consider fine  $\varphi$ -gradings on matrix algebras, which were introduced and classified in [Eld10].

The purpose of this paper is to compute the Weyl groups of all fine gradings on the simple Lie algebras of series  $A$ ,  $B$ ,  $C$  and  $D$ , with the sole exception of type  $D_4$  (which differs from the other types due to the triality phenomenon), over an algebraically closed field  $\mathbb{F}$  of characteristic different from 2. To achieve this, we first determine the automorphisms of each fine  $\varphi$ -grading on the matrix algebra  $\mathcal{R} = M_n(\mathbb{F})$ ,  $n \geq 3$ , and then use the transfer technique of [BK10] to obtain the Weyl group of the corresponding fine grading on the simple Lie algebra  $\mathcal{L} =$

---

2010 *Mathematics Subject Classification.* Primary 17B70, secondary 17B40, 16W50.

*Key words and phrases.* Graded algebra, fine grading, Weyl group, simple Lie algebra.

\* Supported by the Spanish Ministerio de Educación y Ciencia and FEDER (MTM 2010-18370-C04-02) and by the Diputación General de Aragón (Grupo de Investigación de Álgebra).

\*\*Supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada, Discovery Grant # 341792-07.

$[\mathcal{R}, \mathcal{R}]/(Z(\mathcal{R}) \cap [\mathcal{R}, \mathcal{R}])$  or  $\mathcal{K}(\mathcal{R}, \varphi)$ , where in the second case  $\varphi$  is an involution on  $\mathcal{R}$  and  $\mathcal{K}(\mathcal{R}, \varphi)$  stands for the set of skew-symmetric elements with respect to  $\varphi$ .

We adopt the terminology and notation of [EKb], which is recalled in Section 2 for convenience of the reader. In Section 3, we restate the classification of fine  $\varphi$ -gradings on matrix algebras [Eld10] in more explicit terms and determine the relevant automorphism groups of each fine  $\varphi$ -grading (Theorem 3.12). In Section 4, we deal with the simple Lie algebras of series  $A$  (Theorems 4.6 and 4.7) and, in Section 5, with those of series  $B$ ,  $C$  and  $D$  (Theorems 5.6 and 5.7).

## 2. GENERALITIES ON GRADINGS

Let  $\mathcal{A}$  be an algebra (not necessarily associative) over a field  $\mathbb{F}$  and let  $G$  be a group (written multiplicatively).

**Definition 2.1.** A  $G$ -grading on  $\mathcal{A}$  is a vector space decomposition

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

such that

$$\mathcal{A}_g \mathcal{A}_h \subset \mathcal{A}_{gh} \quad \text{for all } g, h \in G.$$

If such a decomposition is fixed, we will refer to  $\mathcal{A}$  as a  $G$ -graded algebra. The nonzero elements  $a \in \mathcal{A}_g$  are said to be *homogeneous of degree  $g$* ; we will write  $\deg a = g$ . The *support* of  $\Gamma$  is the set  $\text{Supp } \Gamma := \{g \in G \mid \mathcal{A}_g \neq 0\}$ .

There are two natural ways to define equivalence relation on graded algebras. We will use the term “isomorphism” for the case when the grading group is a part of definition and “equivalence” for the case when the grading group plays a secondary role. Let

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \text{and} \quad \Gamma' : \mathcal{B} = \bigoplus_{h \in H} \mathcal{B}_h$$

be two gradings on algebras, with supports  $S$  and  $T$ , respectively.

**Definition 2.2.** We say that  $\Gamma$  and  $\Gamma'$  are *equivalent* if there exists an isomorphism of algebras  $\psi : \mathcal{A} \rightarrow \mathcal{B}$  and a bijection  $\alpha : S \rightarrow T$  such that  $\psi(\mathcal{A}_s) = \mathcal{B}_{\alpha(s)}$  for all  $s \in S$ . Any such  $\psi$  will be called an *equivalence* of  $\Gamma$  and  $\Gamma'$  (or of  $\mathcal{A}$  and  $\mathcal{B}$  if the gradings are clear from the context).

The algebras graded by a fixed group  $G$  form a category where the morphisms are the *homomorphisms of  $G$ -graded algebras*, i.e., algebra homomorphisms  $\psi : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\psi(\mathcal{A}_g) \subset \mathcal{B}_g$  for all  $g \in G$ .

**Definition 2.3.** In the case  $G = H$ , we say that  $\Gamma$  and  $\Gamma'$  are *isomorphic* if  $\mathcal{A}$  and  $\mathcal{B}$  are isomorphic as  $G$ -graded algebras, i.e., there exists an isomorphism of algebras  $\psi : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\psi(\mathcal{A}_g) = \mathcal{B}_g$  for all  $g \in G$ .

It is known that if  $\Gamma$  is a grading on a simple Lie algebra, then  $\text{Supp } \Gamma$  generates an abelian group (see e.g. [Koc09, Proposition 3.3]). From now on, we will assume that our grading groups are *abelian*. Given a group grading  $\Gamma$  on an algebra  $\mathcal{A}$ , there are many groups  $G$  such that  $\Gamma$  can be realized as a  $G$ -grading, but there is one distinguished group among them [PZ89].

**Definition 2.4.** Suppose that  $\Gamma$  admits a realization as a  $G_0$ -grading for some group  $G_0$ . We will say that  $G_0$  is a *universal group* of  $\Gamma$  if, for any other realization of  $\Gamma$  as a  $G$ -grading, there exists a unique homomorphism  $G_0 \rightarrow G$  that restricts to identity on  $\text{Supp } \Gamma$ .

One shows that the universal group, which we denote by  $U(\Gamma)$ , exists and depends only on the equivalence class of  $\Gamma$ . Indeed,  $U(\Gamma)$  is generated by  $S = \text{Supp } \Gamma$  with defining relations  $s_1 s_2 = s_3$  whenever  $0 \neq \mathcal{A}_{s_1} \mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$  ( $s_i \in S$ ).

As in [PZ89], we associate to  $\Gamma$  three subgroups of the automorphism group  $\text{Aut}(\mathcal{A})$  as follows.

**Definition 2.5.** The *automorphism group* of  $\Gamma$ , denoted  $\text{Aut}(\Gamma)$ , consists of all automorphisms of  $\mathcal{A}$  that permute the components of  $\Gamma$ . Each  $\psi \in \text{Aut}(\Gamma)$  determines a self-bijection  $\alpha = \alpha(\psi)$  of the support  $S$  such that  $\psi(\mathcal{A}_s) = \mathcal{A}_{\alpha(s)}$  for all  $s \in S$ . The *stabilizer* of  $\Gamma$ , denoted  $\text{Stab}(\Gamma)$ , is the kernel of the homomorphism  $\text{Aut}(\Gamma) \rightarrow \text{Sym}(S)$  given by  $\psi \mapsto \alpha(\psi)$ . Finally, the *diagonal group* of  $\Gamma$ , denoted  $\text{Diag}(\Gamma)$ , is the subgroup of the stabilizer consisting of all automorphisms  $\psi$  such that the restriction of  $\psi$  to any homogeneous component of  $\Gamma$  is the multiplication by a (nonzero) scalar.

Thus  $\text{Aut}(\Gamma)$  is the group of self-equivalences of the graded algebra  $\mathcal{A}$  and  $\text{Stab}(\Gamma)$  is the group of automorphisms of the graded algebra  $\mathcal{A}$ . Also,  $\text{Diag}(\Gamma)$  is isomorphic to the group of characters of  $U(\Gamma)$  via the usual action of characters on  $\mathcal{A}$ : if  $\Gamma$  is a  $G$ -grading (in particular, we may take  $G = U(\Gamma)$ ), then any character  $\chi \in \widehat{G}$  acts as an automorphism of  $\mathcal{A}$  by setting  $\chi * a = \chi(g)a$  for all  $a \in \mathcal{A}_g$  and  $g \in G$ . If  $\dim \mathcal{A} < \infty$ , then  $\text{Diag}(\Gamma)$  is a diagonalizable algebraic group (quasitorus). If, in addition,  $\mathbb{F}$  is algebraically closed and  $\text{char } \mathbb{F} = 0$ , then  $\Gamma$  is the eigenspace decomposition of  $\mathcal{A}$  relative to  $\text{Diag}(\Gamma)$  (see e.g. [Koc09]), the group  $\text{Stab}(\Gamma)$  is the centralizer of  $\text{Diag}(\Gamma)$ , and  $\text{Aut}(\Gamma)$  is its normalizer. If we want to work over an arbitrary field  $\mathbb{F}$ , we can define the subscheme  $\mathbf{Diag}(\Gamma)$  of the automorphism group scheme  $\mathbf{Aut}(\mathcal{A})$  as follows:

$$\mathbf{Diag}(\Gamma)(\mathcal{S}) := \{f \in \text{Aut}_{\mathcal{S}}(\mathcal{A} \otimes \mathcal{S}) \mid f|_{\mathcal{A}_g \otimes \mathcal{S}} \in \mathcal{S}^\times \text{id}_{\mathcal{A}_g \otimes \mathcal{S}} \text{ for all } g \in G\}$$

for any unital commutative associative algebra  $\mathcal{S}$  over  $\mathbb{F}$ . Thus  $\text{Diag}(\Gamma)$  is the group of  $\mathbb{F}$ -points of  $\mathbf{Diag}(\Gamma)$ . One checks that  $\mathbf{Diag}(\Gamma) = U(\Gamma)^D$ , the Cartier dual of  $U(\Gamma)$ , also  $\text{Stab}(\Gamma)$  is the centralizer of  $\mathbf{Diag}(\Gamma)$  and  $\text{Aut}(\Gamma)$  is its normalizer with respect to the action of  $\text{Aut}(\mathcal{A})$  on  $\mathbf{Aut}(\mathcal{A})$  by conjugation (see e.g. [EKa, §2.2]).

**Definition 2.6.** The quotient group  $\text{Aut}(\Gamma)/\text{Stab}(\Gamma)$ , which is a subgroup of  $\text{Sym}(S)$ , will be called the *Weyl group* of  $\Gamma$  and denoted by  $W(\Gamma)$ .

It follows from the universal property of  $U(\Gamma)$  that, for any  $\psi \in \text{Aut}(\Gamma)$ , the bijection  $\alpha(\psi): \text{Supp } \Gamma \rightarrow \text{Supp } \Gamma$  extends to a unique automorphism of  $U(\Gamma)$ . This gives an action of  $\text{Aut}(\Gamma)$  by automorphisms of  $U(\Gamma)$ . Since the kernel of this action is  $\text{Stab}(\Gamma)$ , we may regard  $W(\Gamma) = \text{Aut}(\Gamma)/\text{Stab}(\Gamma)$  as a subgroup of  $\text{Aut}(U(\Gamma))$ . Given a  $G$ -grading  $\Gamma: \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$  and a group homomorphism  $\alpha: G \rightarrow H$ , we obtain the induced  $H$ -grading  ${}^\alpha \Gamma: \mathcal{A} = \bigoplus_{h \in H} \mathcal{A}'_h$  by setting  $\mathcal{A}'_h = \bigoplus_{g \in \alpha^{-1}(h)} \mathcal{A}_g$ . Clearly, an automorphism  $\alpha$  of  $U(\Gamma)$  belongs to  $W(\Gamma)$  if and only if the  $U(\Gamma)$ -gradings  ${}^\alpha \Gamma$  and  $\Gamma$  are isomorphic.

Given gradings  $\Gamma: \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$  and  $\Gamma': \mathcal{A} = \bigoplus_{h \in H} \mathcal{A}'_h$ , we say that  $\Gamma'$  is a *coarsening* of  $\Gamma$ , or that  $\Gamma$  is a *refinement* of  $\Gamma'$ , if for any  $g \in G$  there exists

$h \in H$  such that  $\mathcal{A}_g \subset \mathcal{A}'_h$ . The coarsening (or refinement) is said to be *proper* if the inclusion is proper for some  $g$ . (In particular,  ${}^a\Gamma$  is a coarsening of  $\Gamma$ , which is not necessarily proper.) A grading  $\Gamma$  is said to be *fine* if it does not admit a proper refinement in the class of (abelian) group gradings. Any  $G$ -grading on a finite-dimensional algebra  $\mathcal{A}$  is induced from some fine grading  $\Gamma$  by a homomorphism  $\alpha: U(\Gamma) \rightarrow G$ . The classification of fine gradings on  $\mathcal{A}$  up to equivalence is the same as the classification of maximal diagonalizable subgroupschemes of  $\mathbf{Aut}(\mathcal{A})$  up to conjugation by  $\mathbf{Aut}(\mathcal{A})$  (see e.g. [EKa, §2.2]). Fine gradings on simple Lie algebras belonging to the series  $A$ ,  $B$ ,  $C$  and  $D$  (including  $D_4$ ) were classified in [Eld10] assuming  $\mathbb{F}$  algebraically closed of characteristic 0. If we replace automorphism groups by automorphism group schemes, as was done in [BK10], then the arguments of [Eld10] for all cases except  $D_4$  (which required a completely different method) work under the much weaker assumption — which we adopt from now on — that  $\mathbb{F}$  is *algebraically closed of characteristic different from 2*.

### 3. FINE $\varphi$ -GRADINGS ON MATRIX ALGEBRAS

The goal of this section is to determine certain automorphism groups of fine  $\varphi$ -gradings on matrix algebras. These groups will be used in the next two sections to compute the Weyl groups of fine gradings on simple Lie algebras of series  $A$ ,  $B$ ,  $C$  and  $D$ .

**3.1. Classification of fine  $\varphi$ -gradings on matrix algebras.** Here we present the results of [Eld10, §3] in a more explicit form. We also introduce certain objects that will appear throughout the paper.

**Definition 3.1.** Let  $\mathcal{A}$  be an algebra and let  $\varphi$  be an anti-automorphism of  $\mathcal{A}$ . A  $G$ -grading  $\Gamma: \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$  is said to be a  $\varphi$ -grading if  $\varphi(\mathcal{A}_g) = \mathcal{A}_g$  for all  $g \in G$  (i.e.,  $\varphi$  is an anti-automorphism of the  $G$ -graded algebra  $\mathcal{A}$ ) and  $\varphi^2 \in \text{Diag}(\Gamma)$ . The universal group of a  $\varphi$ -grading is defined disregarding  $\varphi$ .

We have natural concepts of isomorphism and equivalence for  $\varphi$ -gradings. In addition, we will need another relation, which is weaker than equivalence.

**Definition 3.2.** If  $\Gamma_1$  is a  $\varphi_1$ -grading on  $\mathcal{A}$  and  $\Gamma_2$  is a  $\varphi_2$ -gradings on  $\mathcal{B}$ , we will say that  $(\Gamma_1, \varphi_1)$  is *isomorphic* (respectively, *equivalent*) to  $(\Gamma_2, \varphi_2)$  if there exists an isomorphism (respectively, equivalence) of graded algebras  $\psi: \mathcal{A} \rightarrow \mathcal{B}$  such that  $\varphi_2 = \psi\varphi_1\psi^{-1}$ . In the special case  $\mathcal{A} = \mathcal{B}$  and  $\varphi_1 = \varphi_2$ , we will simply say that  $\Gamma_1$  is *isomorphic* (respectively, *equivalent*) to  $\Gamma_2$ . We will say that  $(\Gamma_1, \varphi_1)$  is *weakly equivalent* to  $(\Gamma_2, \varphi_2)$  if there exists an equivalence of graded algebras  $\psi: \mathcal{A} \rightarrow \mathcal{B}$  such that  $\xi\varphi_2 = \psi\varphi_1\psi^{-1}$  for some  $\xi \in \text{Diag}(\Gamma_2)$ .

Note that if  $\varphi$  is an involution, then the condition  $\varphi^2 \in \text{Diag}(\Gamma)$  is satisfied for any  $\Gamma$ . Also, any  $\varphi$ -grading  $\Gamma$  on  $\mathcal{A}$  restricts to the space of skew-symmetric elements  $\mathcal{K}(\mathcal{A}, \varphi)$ .

Suppose  $\mathcal{R}$  is a matrix algebra equipped with a  $G$ -grading  $\Gamma$ . Then  $\mathcal{R}$  is isomorphic to  $\text{End}_{\mathcal{D}}(V)$  where  $\mathcal{D}$  is a matrix algebra with a division grading (i.e., a grading that makes it a graded division algebra) and  $V$  is a graded right  $\mathcal{D}$ -module (which is necessarily free of finite rank). Let  $T \subset G$  be the support of  $\mathcal{D}$ . Then  $T$  is a group and  $\mathcal{D}$  can be identified with a twisted group algebra  $\mathbb{F}^\sigma T$  for some 2-cocycle  $\sigma: T \times T \rightarrow \mathbb{F}^\times$ , i.e.,  $\mathcal{D}$  has a basis  $X_t$ ,  $t \in T$ , such that  $X_u X_v = \sigma(u, v) X_{uv}$  for all

$u, v \in T$  (we may assume  $X_e = I$ , the identity element of  $\mathcal{D}$ ). Let  $\beta(u, v) = \frac{\sigma(u, v)}{\sigma(v, u)}$ , so

$$X_u X_v = \beta(u, v) X_v X_u.$$

Then  $\beta: T \times T \rightarrow \mathbb{F}^\times$  is a nondegenerate alternating bicharacter — see e.g. [BK10, §2]. A division grading on a matrix algebra with a given support  $T$  and bicharacter  $\beta$  can be constructed as follows. Since  $\beta$  is nondegenerate and alternating,  $T$  admits a “symplectic basis”, i.e., there exists a decomposition of  $T$  into the direct product of cyclic subgroups:

$$T = (H'_1 \times H''_1) \times \cdots \times (H'_r \times H''_r)$$

such that  $H'_i \times H''_i$  and  $H'_j \times H''_j$  are  $\beta$ -orthogonal for  $i \neq j$ , and  $H'_i$  and  $H''_i$  are in duality by  $\beta$ . Denote by  $\ell_i$  the order of  $H'_i$  and  $H''_i$ . (We may assume without loss of generality that  $\ell_i$  are prime powers.) If we pick generators  $a_i$  and  $b_i$  for  $H'_i$  and  $H''_i$ , respectively, then  $\varepsilon_i := \beta(a_i, b_i)$  is a primitive  $\ell_i$ -th root of unity, and all other values of  $\beta$  on the elements  $a_1, b_1, \dots, a_r, b_r$  are 1. Define the following elements of the algebra  $M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$ :

$$X_{a_i} = I \otimes \cdots \otimes I \otimes X_i \otimes I \otimes \cdots \otimes I \quad \text{and} \quad X_{b_i} = I \otimes \cdots \otimes I \otimes Y_i \otimes I \otimes \cdots \otimes I,$$

where

$$X_i = \begin{bmatrix} \varepsilon_i^{n-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_i^{n-2} & 0 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & \varepsilon_i & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad Y_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

are the generalized Pauli matrices in the  $i$ -th factor,  $M_{\ell_i}(\mathbb{F})$ . Finally, set

$$X_{(a_1^{i_1}, b_1^{j_1}, \dots, a_r^{i_r}, b_r^{j_r})} = X_{a_1}^{i_1} X_{b_1}^{j_1} \cdots X_{a_r}^{i_r} X_{b_r}^{j_r}.$$

Identify  $M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$  with  $M_\ell(\mathbb{F})$ ,  $\ell = \ell_1 \cdots \ell_r = \sqrt{|T|}$ , via Kronecker product. Then

$$M_\ell(\mathbb{F}) = \bigoplus_{t \in T} \mathbb{F} X_t$$

is a division grading with support  $T$  and bicharacter  $\beta$ .

Let  $\varphi$  be an anti-automorphism of  $\mathcal{R}$  such that  $\Gamma$  is a  $\varphi$ -grading. It is shown in [Eld10, §3] that there exists an involution  $\varphi_0$  of the graded algebra  $\mathcal{D}$  and a  $\varphi_0$ -sesquilinear form  $B: V \times V \rightarrow \mathcal{D}$ , which is nondegenerate, homogeneous and balanced, such that, for all  $r \in \mathcal{R}$ ,  $\varphi(r)$  is the adjoint of  $r$  with respect to  $B$ , i.e.,  $B(x, \varphi(r)y) = B(rx, y)$  for all  $x, y \in V$  and  $r \in \mathcal{R}$ . By  $\varphi_0$ -sesquilinear we mean that  $B$  is  $\mathbb{F}$ -bilinear and, for all  $x, y \in V$  and  $d \in \mathcal{D}$ , we have  $B(xd, y) = \varphi_0(d)B(x, y)$  and  $B(x, yd) = B(x, y)d$ ; by *balanced* we mean that, for all homogeneous  $x, y \in V$ ,  $B(x, y) = 0$  is equivalent to  $B(y, x) = 0$ . Moreover, the existence of  $\varphi_0$  forces  $T$  to be an elementary 2-group. The pair  $(\varphi_0, B)$  is uniquely determined by  $\varphi$  up to the following transformations: for any nonzero homogeneous  $d \in \mathcal{D}$ , we may simultaneously replace  $\varphi_0$  by  $\varphi'_0: a \mapsto d\varphi_0(a)d^{-1}$  and  $B$  by  $B' = dB$ . Using Pauli matrices (of order 2) as above to construct a realization of  $\mathcal{D}$ , we see that matrix transpose  $X \mapsto {}^t X$  preserves the grading: for any  $u \in T$ , the transpose of  $X_u$  equals  $\pm X_u$ . It follows from [BK10, Proposition 2.3] that  $(\varphi_0, B)$  can be adjusted so that  $\varphi_0$  coincides with the matrix transpose. We will always assume that  $(\varphi_0, B)$

is adjusted in this way, which makes  $B$  unique up to a scalar in  $\mathbb{F}$ . Also, we may write

$$\varphi_0(X_u) = \beta(u)X_u$$

where  $\beta(u) \in \{\pm 1\}$  for all  $u \in T$ . If we regard  $T$  as a vector space over the field of two elements, then the function  $\beta: T \rightarrow \{\pm 1\}$  is a quadratic form whose polar form is the bicharacter  $\beta: T \times T \rightarrow \{\pm 1\}$ .

We will say that a  $\varphi$ -grading is *fine* if it is not a proper coarsening of another  $\varphi$ -grading. The following construction of fine  $\varphi$ -gradings on matrix algebras was given in [Eld10] starting from  $\mathcal{D}$ . We start from  $T$ , an elementary 2-group of even dimension, i.e.,  $T = \mathbb{Z}_2^{\dim T}$ , which we continue to write multiplicatively. Let  $\beta$  be a nondegenerate alternating bicharacter on  $T$ . Fix a realization,  $\mathcal{D}$ , of the matrix algebra endowed with a division grading with support  $T$  and bicharacter  $\beta$ , and let  $\varphi_0$  be the matrix transpose on  $\mathcal{D}$ . Let  $q \geq 0$  and  $s \geq 0$  be two integers. Let

$$(1) \quad \tau = (t_1, \dots, t_q), \quad t_i \in T.$$

Denote by  $\tilde{G} = \tilde{G}(T, q, s, \tau)$  the abelian group generated by  $T$  and the symbols  $\tilde{g}_1, \dots, \tilde{g}_{q+2s}$  with defining relations

$$(2) \quad \tilde{g}_1^2 t_1 = \dots = \tilde{g}_q^2 t_q = \tilde{g}_{q+1} \tilde{g}_{q+2} = \dots = \tilde{g}_{q+2s-1} \tilde{g}_{q+2s}.$$

**Definition 3.3.** Let  $\mathcal{M}(\mathcal{D}, q, s, \tau)$  be the  $\tilde{G}$ -graded algebra  $\text{End}_{\mathcal{D}}(V)$  where  $V$  has a  $\mathcal{D}$ -basis  $\{v_1, \dots, v_{q+2s}\}$  with  $\deg v_i = \tilde{g}_i$ . Let  $n = (q+2s)2^{\frac{1}{2}\dim T}$  and  $\mathcal{R} = M_n(\mathbb{F})$ . The grading  $\Gamma$  on  $\mathcal{R}$  obtained by identifying  $\mathcal{R}$  with  $\mathcal{M}(\mathcal{D}, q, s, \tau)$  will be denoted by  $\Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$ . In other words, we define this grading by identifying  $\mathcal{R} = M_{q+2s}(\mathcal{D})$  and setting  $\deg(E_{ij} \otimes X_t) := \tilde{g}_i t \tilde{g}_j^{-1}$ . By abuse of notation, we will also write  $\Gamma_{\mathcal{M}}(T, q, s, \tau)$ .

Let  $\tilde{G}^0$  be the subgroup of  $\tilde{G}$  generated by  $\text{Supp } \Gamma$ , which consists of the elements  $z_{i,j,t} := \tilde{g}_i t \tilde{g}_j^{-1}$ ,  $t \in T$  (so  $z_{i,i,t} = t$  for all  $t \in T$ ). Set  $z_i := z_{i,i+1,e}$  for  $i = 1, \dots, q$  ( $i \neq q$  if  $s = 0$ ),  $z_{q+i} = z_{q+2i-1,q+2i+1,e}$  for  $i = 1, \dots, s-1$ , and  $z_{q+s} = z_{q+1,q+2,e}$  (if  $s > 0$ ). If  $s = 0$ , then  $\tilde{G}^0$  is generated by  $T$  and the elements  $z_1, \dots, z_{q-1}$ . If  $s = 1$ , then  $\tilde{G}^0$  is generated by  $T$  and  $z_1, \dots, z_{q+1}$ . If  $s > 1$ , then relations (2) imply that  $z_{q+2i,q+2i+2,e} = z_{q+i}^{-1}$  for  $i = 1, \dots, s-1$ , hence  $\tilde{G}^0$  is generated by  $T$  and  $z_1, \dots, z_{q+s}$ . Moreover, relations (2) are equivalent to the following:

$$z_i^2 = t_i t_{i+1} \quad (1 \leq i < q), \quad z_q^2 z_{q+s} = t_q \quad (\text{if } q > 0 \text{ and } s > 0).$$

Let  $\tilde{G}^1$  be the subgroup generated by  $T$  and  $z_1, \dots, z_{q-1}$ . Let  $\tilde{G}^2$  be the subgroup generated by  $z_1, \dots, z_s$  if  $q = 0$  and by  $z_q, \dots, z_{q+s-1}$  if  $q > 0$ . Then it is clear from the above relations that  $\tilde{G}^0 = \tilde{G}^1 \times \tilde{G}^2$ ,  $\tilde{G}^2 \cong \mathbb{Z}^s$ , while  $\tilde{G}^1 = T$  if  $q = 0$  and  $\tilde{G}^1 \cong \mathbb{Z}_2^{\dim T + q - 1 - 2 \dim T_0} \times \mathbb{Z}_4^{\dim T_0}$  if  $q > 0$ , where  $T_0$  is the subgroup of  $T$  generated by the elements  $t_i t_{i+1}$ ,  $i = 1, \dots, q-1$ . To summarize:

$$(3) \quad \tilde{G}^0 \cong \mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0, q-1)} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s.$$

Note that relations (2) are also equivalent to the following:

$$\begin{aligned} z_{i,j,t_i t} &= z_{j,i,t_j t}, & i, j \leq q, \quad t \in T; \\ z_{i,q+2j-1,t_i t} &= z_{q+2j,i,t}, \quad z_{i,q+2j,t_i t} = z_{q+2j-1,i,t}, & i \leq q, \quad j \leq s, \quad t \in T; \\ z_{q+2i-1,q+2j-1,t} &= z_{q+2j,q+2i,t}, & i, j \leq s, \quad t \in T; \\ z_{q+2i-1,q+2j,t} &= z_{q+2j-1,q+2i,t}, & i, j \leq s, \quad i \neq j, \quad t \in T. \end{aligned}$$

One verifies that, apart from the above equalities and  $z_{i,i,t} = t$ , the elements  $z_{i,j,t}$  are distinct, so the support of  $\Gamma = \Gamma_{\mathcal{M}}(\tilde{G}, \mathcal{D}, \kappa, \tilde{\gamma})$  is given by

$$\begin{aligned} \text{Supp } \Gamma = & \{z_{i,j,t} \mid i < j \leq q, t \in T\} \cup \{z_{i,q+j,t} \mid i \leq q, j \leq 2s, t \in T\} \\ & \cup \{z_{q+2i-1,q+2j-1,t} \mid i < j \leq s, t \in T\} \cup \{z_{q+2i,q+2j,t} \mid i < j \leq s, t \in T\} \\ & \cup \{z_{q+2i-1,q+2j,t} \mid i, j \leq s, i \neq j, t \in T\} \\ & \cup \{z_{q+2i-1,q+2i,t} \mid i \leq s, t \in T\} \cup \{z_{q+2i,q+2i-1,t} \mid i \leq s, t \in T\} \cup T, \end{aligned}$$

where the union is disjoint and all homogeneous components except those that appear in the last line have dimension 2, the components of degrees  $z_{q+2i-1,q+2i,t}$  and  $z_{q+2i,q+2i-1,t}$  have dimension 1, and the components of degree  $t$  have dimension  $q + 2s$ .

**Proposition 3.4.** *Let  $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$ . Then  $\tilde{G}^0 = \tilde{G}^0(T, q, s, \tau)$  is the universal group of  $\Gamma$ , and  $\text{Diag}(\Gamma)$  consists of all automorphisms of the form  $X \mapsto DXD^{-1}$ ,  $X \in \mathcal{R}$ , where*

$$(4) \quad D = \text{diag}(\lambda_1, \dots, \lambda_{q+2s}) \otimes X_t, \quad \lambda_i \in \mathbb{F}^\times, t \in T,$$

satisfying the relation

$$(5) \quad \lambda_1^2 \beta(t, t_1) = \dots = \lambda_q^2 \beta(t, t_q) = \lambda_{q+1} \lambda_{q+2} = \dots = \lambda_{q+2s-1} \lambda_{q+2s}.$$

*Proof.* The relations  $z_{i,\ell,u} z_{\ell,j,v} = z_{i,j,uv}$ ,  $u, v \in T$ , can be rewritten in terms of the elements of  $\text{Supp } \Gamma$ , producing a set of defining relations for  $\tilde{G}^0$ . It follows that  $\tilde{G}^0$  is the universal group of  $\Gamma$ .

Since  $\tilde{G}^0$  is the universal group of  $\Gamma$ ,  $\text{Diag}(\Gamma)$  consists of all automorphisms of the form  $X \mapsto \chi * X$  where  $\chi$  is a character of  $\tilde{G}^0$ . Since  $\mathbb{F}^\times$  is a divisible group, we can assume that  $\chi$  is a character of  $\tilde{G}$ . Let  $\lambda_i = \chi(\tilde{g}_i)$ ,  $i = 1, \dots, q + 2s$ . Let  $t$  be the element of  $T$  such that  $\chi(u) = \beta(t, u)$  for all  $u \in T$ . Looking at relations (2), we see that (5) must hold. Conversely, any  $t \in T$  and a set of  $\lambda_i \in \mathbb{F}^\times$  satisfying (5) will determine a character  $\chi$  of  $\tilde{G}$ . It remains to observe that the action of  $\chi$  on  $\mathcal{R}$  coincides with the conjugation by  $D$  as in (4).  $\square$

The following is Proposition 3.3 from [Eld10].

**Theorem 3.5.** *Consider the grading  $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$  on  $\mathcal{R} = M_{q+2s}(\mathcal{D})$  by  $\tilde{G}^0 = \tilde{G}^0(T, q, s, \tau)$  where  $\tau$  is given by (1). Let  $\mu = (\mu_1, \dots, \mu_s)$  where  $\mu_i$  are scalars in  $\mathbb{F}^\times$ . Let  $\varphi = \varphi_{\tau, \mu}$  be the anti-automorphism of  $\mathcal{R}$  defined by  $\varphi(X) = \Phi^{-1}({}^t X) \Phi$ ,  $X \in \mathcal{R}$ , where  $\Phi$  is the block-diagonal matrix given by*

$$(6) \quad \Phi = \text{diag} \left( X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ \mu_1 I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ \mu_s I & 0 \end{bmatrix} \right)$$

and  $I$  is the identity element of  $\mathcal{D}$ . Then  $\Gamma$  is a fine  $\varphi$ -grading unless  $q = 2$ ,  $s = 0$  and  $t_1 = t_2$ . In the latter case,  $\Gamma$  can be refined to a  $\varphi$ -grading that makes  $\mathcal{R}$  a graded division algebra.  $\square$

This result and the discussion preceding Proposition 3.8 in [Eld10] yield

**Theorem 3.6.** *Let  $\Gamma$  be a fine  $\varphi$ -grading on the matrix algebra  $\mathcal{R} = M_n(\mathbb{F})$  over an algebraically closed field  $\mathbb{F}$ ,  $\text{char } \mathbb{F} \neq 2$ . Then  $(\Gamma, \varphi)$  is equivalent to some  $(\Gamma_{\mathcal{M}}(T, q, s, \tau), \varphi_{\tau, \mu})$  as in Theorem 3.5 where  $(q + 2s)2^{\frac{1}{2} \dim T} = n$ .  $\square$*

In [Eld10], in order to obtain the classification of fine gradings on simple Lie algebras of series  $A$ , one classifies, *up to weak equivalence*, all pairs  $(\Gamma, \varphi)$  where  $\Gamma$  is a fine  $\varphi$ -grading on a matrix algebra. At the same time, for series  $B$ ,  $C$  and  $D$ , one classifies, *up to equivalence*, such pairs where  $\varphi$  is an *involution* of appropriate type: orthogonal for series  $B$  and  $D$  (we write  $\text{sgn}(\varphi) = 1$ ) and symplectic for series  $C$  (we write  $\text{sgn}(\varphi) = -1$ ). The classifications involve equivalences  $\mathcal{D} \rightarrow \mathcal{D}'$  satisfying certain conditions, where  $\mathcal{D}$  and  $\mathcal{D}'$  are matrix algebras with division gradings. If  $T$  is the support of  $\mathcal{D}$  and  $T'$  is the support of  $\mathcal{D}'$ , then the graded algebras  $\mathcal{D}$  and  $\mathcal{D}'$  are equivalent if and only if the groups  $T$  and  $T'$  are isomorphic. Identifying  $T$  and  $T'$ , we may assume that  $\mathcal{D} = \mathcal{D}'$  and look at self-equivalences of  $\mathcal{D}$ , i.e., the elements of  $\text{Aut}(\Gamma_0)$  where  $\Gamma_0$  is the grading on  $\mathcal{D}$ . By [EKb, Proposition 2.7], the Weyl group  $W(\Gamma_0)$  is isomorphic to  $\text{Aut}(T, \beta)$ , the group of automorphisms of  $T$  that preserve the bicharacter  $\beta$ . Explicitly, if  $\psi_0 \in \text{Aut}(\Gamma_0)$ , then  $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$ , for all  $t \in T$ , where  $\alpha \in \text{Aut}(T, \beta)$ , and the mapping  $\psi_0 \mapsto \alpha$  yields an isomorphism  $\text{Aut}(\Gamma_0)/\text{Stab}(\Gamma_0) \rightarrow \text{Aut}(T, \beta)$ . Hence the conditions in [Eld10] can be rewritten in terms of the group  $T$  rather than the graded division algebra  $\mathcal{D}$ . Note that  $\text{Aut}(T, \beta)$  can be regarded as a sort of symplectic group; in particular, if  $T$  is an elementary 2-group, then  $\text{Aut}(T, \beta) \cong \text{Sp}_m(2)$  where  $m = \dim T$ .

**Definition 3.7.** Given  $\tau$  as in (1), we will denote by  $\Sigma(\tau)$  the multiset in  $T$  determined by  $\tau$ , i.e., the underlying set of  $\Sigma(\tau)$  consists of the elements that occur in  $(t_1, \dots, t_q)$ , and the multiplicity of each element is the number of times it occurs there.

The group  $\text{Aut}(T, \beta)$  acts naturally on  $T$ , so we can form the semidirect product  $T \rtimes \text{Aut}(T, \beta)$ , which also acts on  $T$ : a pair  $(u, \alpha)$  sends  $t \in T$  to  $\alpha(t)u$ . Clearly, if  $\dim T = 2r$ , then  $T \rtimes \text{Aut}(T, \beta)$  is isomorphic to  $\text{ASp}_{2r}(2)$ , the affine symplectic group of order  $2r$  over the field of two elements (“rigid motions” of the symplectic space of dimension  $2r$ ).

Using this notation, Theorem 3.17 of [Eld10] can be recast as follows:

**Theorem 3.8.** *Consider two pairs,  $(\Gamma, \varphi)$  and  $(\Gamma', \varphi')$ , as in Theorem 3.5, namely,  $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$ ,  $\varphi = \varphi_{\tau, \mu}$  and  $\Gamma' = \Gamma_{\mathcal{M}}(T', q', s', \tau')$ ,  $\varphi' = \varphi_{\tau', \mu'}$ , where  $T = \mathbb{Z}_2^{2r}$  and  $T' = \mathbb{Z}_2^{2r'}$ . Then  $(\Gamma, \varphi)$  and  $(\Gamma', \varphi')$  are weakly equivalent if and only if  $r = r'$ ,  $q = q'$ ,  $s = s'$ , and  $\Sigma(\tau)$  is conjugate to  $\Sigma(\tau')$  by the natural action of  $T \rtimes \text{Aut}(T, \beta) \cong \text{ASp}_{2r}(2)$ .  $\square$*

Let  $\psi_0: \mathcal{D} \rightarrow \mathcal{D}$  be an equivalence. Then the map  $\psi_0^{-1}\varphi_0\psi_0$  is an involution of the graded algebra  $\mathcal{D}$ , which has the same type as  $\varphi_0$  (orthogonal). Hence there exists a nonzero homogeneous element  $d_0 \in \mathcal{D}$  such that

$$(7) \quad d_0\varphi_0(d)d_0^{-1} = (\psi_0^{-1}\varphi_0\psi_0)(d) \quad \text{for all } d \in \mathcal{D}.$$

Note that  $d_0$  is determined up to a scalar in  $\mathbb{F}$ . Moreover,  $d_0$  is symmetric with respect to  $\varphi_0$ . By a similar argument,  $\psi_0(d_0)$  is also symmetric. Let  $\alpha$  be the element of  $\text{Aut}(T, \beta)$  corresponding to  $\psi_0$  and let  $t_0$  be the degree of  $d_0$ . Then (7) is equivalent to the following:

$$(8) \quad \beta(t_0, t)\beta(t) = \beta(\alpha(t)) \quad \text{for all } t \in T,$$

so  $t_0$  depends only on  $\alpha$ . Moreover,  $\beta(t_0) = \beta(\alpha(t_0)) = 1$ .

**Definition 3.9.** For any  $\alpha \in \text{Aut}(T, \beta)$ , the map  $t \mapsto \beta(\alpha^{-1}(t))\beta(t)$  is a character of  $T$ , so there exists a unique element  $t_\alpha \in T$  such that  $\beta(t_\alpha, t) = \beta(\alpha^{-1}(t))\beta(t)$  for all  $t \in T$ . We define a new action of the group  $\text{Aut}(T, \beta)$  on  $T$  by setting

$$\alpha \cdot t := \alpha(t)t_\alpha \quad \text{for all } \alpha \in \text{Aut}(T, \beta) \text{ and } t \in T.$$

In other words,  $\text{Aut}(T, \beta)$  acts through the (injective) homomorphism to  $T \rtimes \text{Aut}(T, \beta)$ ,  $\alpha \mapsto (t_\alpha, \alpha)$ , and the natural action of  $T \rtimes \text{Aut}(T, \beta)$  on  $T$ .

Comparing this definition with equation (8), which defines the element  $t_0$  associated to  $\alpha$ , we see that  $t_\alpha = \alpha(t_0)$ . In particular,  $\beta(t_\alpha) = 1$ . This implies that  $\beta(\alpha \cdot t) = \beta(t)$  for all  $t \in T$ , so the sets

$$T_+ := \{t \in T \mid \beta(t) = 1\} \quad \text{and} \quad T_- := \{t \in T \mid \beta(t) = -1\},$$

which correspond, respectively, to symmetric and skew-symmetric homogeneous components of  $\mathcal{D}$  (relative to  $\varphi_0$ ), are invariant under the twisted action of  $\text{Aut}(T, \beta)$ .

Now Proposition 3.8(2) and Theorem 3.22 of [Eld10] can be recast as follows:

**Theorem 3.10.** *Let  $\varphi = \varphi_{\tau, \mu}$  be as in Theorem 3.5. Then  $\varphi$  is an involution with  $\text{sgn}(\varphi) = \delta$  if and only if*

$$\delta = \beta(t_1) = \dots = \beta(t_q) = \mu_1 = \dots = \mu_s.$$

*For gradings  $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$  with  $T = \mathbb{Z}_2^{2r}$  and  $\Gamma' = \Gamma_{\mathcal{M}}(T', q', s', \tau')$  with  $T' = \mathbb{Z}_2^{2r'}$  and for involutions  $\varphi = \varphi_{\tau, \mu}$  and  $\varphi' = \varphi_{\tau', \mu'}$ , the pairs  $(\Gamma, \varphi)$  and  $(\Gamma', \varphi')$  are equivalent if and only if  $r = r'$ ,  $q = q'$ ,  $s = s'$ ,  $\text{sgn}(\varphi) = \text{sgn}(\varphi')$ , and  $\Sigma(\tau)$  is conjugate to  $\Sigma(\tau')$  by the twisted action of  $\text{Aut}(T, \beta) \cong \text{Sp}_{2r}(2)$  as in Definition 3.9.*  $\square$

**3.2. Automorphism groups of fine  $\varphi$ -gradings on matrix algebras.** We are now going to study automorphisms of the fine  $\varphi$ -gradings  $\Gamma_{\mathcal{M}}(T, q, s, \tau)$ . We begin with some general observations. Let  $\mathcal{D}$  and  $\mathcal{D}'$  be graded division algebras, with the same grading group  $G$ . Let  $V$  be a graded right  $\mathcal{D}$ -module and  $V'$  a graded right  $\mathcal{D}'$ -module, both of nonzero finite rank. By an *isomorphism from  $(\mathcal{D}, V)$  to  $(\mathcal{D}', V')$*  we mean a pair  $(\psi_0, \psi_1)$  where  $\psi_0: \mathcal{D} \rightarrow \mathcal{D}'$  is an isomorphism of graded algebras,  $\psi_1: V \rightarrow V'$  is an isomorphism of graded vector spaces over  $\mathbb{F}$ , and  $\psi_1(vd) = \psi_1(v)\psi_0(d)$  for all  $v \in V$  and  $d \in \mathcal{D}$ .

Let  $\mathcal{R} = \text{End}_{\mathcal{D}}(V)$  and  $\mathcal{R}' = \text{End}_{\mathcal{D}'}(V')$ . If  $\psi: \mathcal{R} \rightarrow \mathcal{R}'$  is an isomorphism of graded algebras, then there exist  $g \in G$  and an isomorphism  $(\psi_0, \psi_1)$  from  $(\mathcal{D}, V^{[g]})$  to  $(\mathcal{D}', V')$  such that  $\psi_1(rv) = \psi(r)\psi_1(v)$  for all  $r \in \mathcal{R}$  and  $v \in V$  (see e.g. [Eld10, Proposition 2.5]). Here  $V^{[g]}$  denotes a shift of grading: the  $(\mathcal{R}, \mathcal{D})$ -bimodule structure of  $V^{[g]}$  is the same as that of  $V$ , but we set  $V_h^{[g]} = V_{hg^{-1}}$  for all  $h \in G$ . Conversely, given an isomorphism  $(\psi_0, \psi_1)$  of the above pairs, there exists a unique isomorphism  $\psi: \mathcal{R} \rightarrow \mathcal{R}'$  of graded algebras such that  $\psi_1(rv) = \psi(r)\psi_1(v)$  for all  $r \in \mathcal{R}$  and  $v \in V$ . Two isomorphisms  $(\psi_0, \psi_1)$  and  $(\psi'_0, \psi'_1)$  determine the same isomorphism  $\mathcal{R} \rightarrow \mathcal{R}'$  if and only if there exists a nonzero homogeneous  $d \in \mathcal{D}'$  such that  $\psi'_0(x) = d^{-1}\psi_0(x)d$  and  $\psi'_1(v) = \psi_1(v)d$  for all  $x \in \mathcal{D}$  and  $v \in V$ .

**Lemma 3.11.** *Let  $\psi: \mathcal{R} \rightarrow \mathcal{R}'$  be the isomorphism of graded algebras determined by an isomorphism  $(\psi_0, \psi_1)$  from  $(\mathcal{D}, V^{[g]})$  to  $(\mathcal{D}', V')$ . Suppose that the graded algebras  $\mathcal{R}$  and  $\mathcal{R}'$  admit anti-automorphisms  $\varphi$  and  $\varphi'$ , respectively, determined by a  $\varphi_0$ -sesquilinear form  $B: V \times V \rightarrow \mathcal{D}$  and a  $\varphi'_0$ -sesquilinear form  $B': V' \times V' \rightarrow \mathcal{D}'$ .*

Then  $\varphi' = \psi\varphi\psi^{-1}$  if and only if there exists a nonzero homogeneous  $d_0 \in \mathcal{D}$  such that

$$(9) \quad B'(\psi_1(v), \psi_1(w)) = \psi_0(d_0 B(v, w)) \quad \text{for all } v, w \in V.$$

Moreover,  $d_0\varphi_0(d)d_0^{-1} = (\psi_0^{-1}\varphi'_0\psi_0)(d)$  for all  $d \in \mathcal{D}$ .

*Proof.* Set  $\varphi'' := \psi^{-1}\varphi'\psi$  and  $B''(v, w) := \psi_0^{-1}(B'(\psi_1(v), \psi_1(w)))$  for all  $v, w \in V$ . Then we compute:

$$\begin{aligned} B''(v, wd) &= \psi_0^{-1}(B'(\psi_1(v), \psi_1(w)\psi_0(d))) \\ &= \psi_0^{-1}(B'(\psi_1(v), \psi_1(w))\psi_0(d)) = B''(v, w)d; \\ B''(vd, w) &= \psi_0^{-1}(B'(\psi_1(v)\psi_0(d), \psi_1(w))) \\ &= \psi_0^{-1}(\varphi'_0(\psi_0(d))B'(\psi_1(v), \psi_1(w))) = (\psi_0^{-1}\varphi'_0\psi_0)(d)B''(v, w); \\ B''(v, \varphi''(r)w) &= \psi_0^{-1}(B'(\psi_1(v), \psi(\varphi''(r))\psi_1(w))) \\ &= \psi_0^{-1}(B'(\psi_1(v), \varphi'(\psi(r))\psi_1(w))) \\ &= \psi_0^{-1}(B'(\psi(r)\psi_1(v), \psi_1(w))) = B''(rv, w). \end{aligned}$$

We have shown that  $B''$  is a  $(\psi_0^{-1}\varphi'_0\psi_0)$ -sesquilinear form corresponding to  $\varphi''$ . Hence  $\varphi'' = \varphi$  if and only if there exists a nonzero homogeneous element  $d_0 \in \mathcal{D}$  such that  $B'' = d_0 B$ , i.e., equation (9) holds.  $\square$

Now consider  $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$  and  $\varphi = \varphi_{\tau, \mu}$  as in Theorem 3.5. There are two kinds of automorphism groups that we will need. Namely, there is

$$\text{Aut}^*(\Gamma, \varphi) := \{\psi \in \text{Aut}(\Gamma) \mid \psi\varphi\psi^{-1} = \xi\varphi \text{ for some } \xi \in \text{Diag}(\Gamma)\},$$

which will be relevant to computing the Weyl group of the corresponding fine grading on the simple Lie algebra of type  $A$ , and there is

$$\text{Aut}(\Gamma, \varphi) := \{\psi \in \text{Aut}(\Gamma) \mid \psi\varphi\psi^{-1} = \varphi\},$$

which will be relevant to computing the Weyl groups of fine gradings on the simple Lie algebras of types  $B$ ,  $C$  and  $D$ . Hence, we are interested in  $\text{Aut}(\Gamma, \varphi)$  only if  $\varphi$  is an involution. Similarly, define

$$\text{Stab}(\Gamma, \varphi) := \{\psi \in \text{Stab}(\Gamma) \mid \psi\varphi\psi^{-1} = \varphi\}.$$

(We could also define  $\text{Stab}^*(\Gamma, \varphi)$ , but we will not need it.)

Recall that  $\Gamma$  is the grading on  $\mathcal{R} = \text{End}_{\mathcal{D}}(V)$  where  $\mathcal{D}$  is a matrix algebra equipped with a division grading with support  $T = \mathbb{Z}_2^{2r}$  and bicharacter  $\beta$ , and  $V$  has a  $\mathcal{D}$ -basis  $\{v_1, \dots, v_k\}$  with  $\deg v_i = \tilde{g}_i$  and  $k = q + 2s$ . We will use the universal group  $\tilde{G}^0$  for the grading  $\Gamma$ . If  $\psi: \mathcal{R} \rightarrow \mathcal{R}$  is an equivalence, then there exists an automorphism  $\alpha$  of the group  $\tilde{G}^0$  such that  $\psi$  sends  ${}^\alpha\Gamma$  to  $\Gamma$ . In other words,  $\psi: \mathcal{R}' \rightarrow \mathcal{R}$  is an isomorphism of graded algebras where  $\mathcal{R}'$  is  $\mathcal{R}$  as an algebra, but equipped with the grading  ${}^\alpha\Gamma$ . Define  $\mathcal{D}'$  similarly to  $\mathcal{R}'$ , using the restriction of  $\alpha$  to  $T \subset \tilde{G}^0$ . The support of  $\mathcal{D}'$  is  $T' = \alpha(T)$ . Since  $V^{[\tilde{g}_1^{-1}]}$  is  $\tilde{G}^0$ -graded, we can also define  $V'$  so that  $\mathcal{R}' = \text{End}_{\mathcal{D}'}(V')$  as a graded algebra. Therefore,  $\psi$  is determined by  $(\psi_0, \psi_1)$  where  $\psi_0: \mathcal{D}' \rightarrow \mathcal{D}$  is an isomorphism of graded algebras and  $\psi_1: V' \rightarrow V$  is an isomorphism up to a shift of grading. Hence  $T' = T$  and  $\psi_0 \in \text{Aut}(\Gamma_0)$ , so  $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$ , for all  $t \in T$ , and the map  $\alpha: T \rightarrow T$  belongs

to  $\text{Aut}(T, \beta) \cong \text{Sp}_{2r}(2)$ . Also, if  $\Psi$  is the matrix of  $\psi_1$  relative to  $\{v_1, \dots, v_k\}$ , we have

$$\psi(X) = \Psi \psi_0(X) \Psi^{-1} \quad \text{for all } X \in \mathcal{R}.$$

Since all  $\tilde{g}_i$  are distinct modulo  $T$ , matrix  $\Psi$  necessarily has the form  $\Psi = PD$  where  $P$  is a permutation matrix and  $D = \text{diag}(d_1, \dots, d_k)$  where  $d_i$  are nonzero homogeneous elements of  $\mathcal{D}$ . Moreover, the permutation  $\pi \in \text{Sym}(k)$  corresponding to  $P$  and the coset of  $\psi_0$  modulo  $\text{Stab}(\Gamma_0)$  are uniquely determined by  $\psi$ . Hence, we have a well-defined homomorphism

$$\text{Aut}(\Gamma) \rightarrow \text{Sym}(k) \times \text{Aut}(T, \beta)$$

that sends  $\psi$  to the corresponding  $(\pi, \alpha)$ .

Now we turn to the anti-automorphism  $\varphi: \mathcal{R} \rightarrow \mathcal{R}$ , which is given by the adjoint with respect to a  $\varphi_0$ -sesquilinear form  $B$  on  $V$  where  $\varphi_0: \mathcal{D} \rightarrow \mathcal{D}$  is given by matrix transpose,  $X_t \mapsto \beta(t)X_t$  for all  $t \in T$ . Recall that such  $B$  is determined up to a scalar in  $\mathbb{F}$ . We can take for  $B$  the  $\varphi_0$ -sesquilinear form whose matrix with respect to  $\{v_1, \dots, v_k\}$  is  $\Phi$  displayed in Theorem 3.5. Pick  $\xi \in \text{Diag}(\Gamma)$  and let  $B'$  be a  $\varphi_0$ -sesquilinear form on  $V$  corresponding to  $\xi\varphi$ . By Lemma 3.11,  $\psi$  satisfies  $\psi\varphi\psi^{-1} = \xi\varphi$  if and only if condition (9) holds for some nonzero homogeneous  $d_0 \in \mathcal{D}$ . Clearly, (9) is equivalent to (7) and

$$(10) \quad \widehat{\Phi} = \psi_0(d_0\Phi),$$

where  $\widehat{\Phi}$  is the matrix of  $B'$  relative to  $\{\psi_1(v_1), \dots, \psi_1(v_k)\}$ . Recall that (7) is equivalent to condition (8) on  $t_0 := \deg d_0$ . To summarize,  $\psi$  satisfies  $\psi\varphi\psi^{-1} = \xi\varphi$  if and only if

$$(11) \quad \widehat{\Phi} = d_0\psi_0(\Phi)$$

for some  $d_0 \in \mathcal{D}$  of degree  $t_\alpha$  as in Definition 3.9 (we have replaced  $\psi_0(d_0)$  in (10) by  $d_0$  to simplify notation).

The matrix of  $B'$  relative to  $\{v_1, \dots, v_k\}$  is  $\Phi(D')^{-1}$  where  $\xi(X) = D'X(D')^{-1}$ , for all  $X \in \mathcal{R}$ , with  $D'$  of the form given by Proposition 3.4:  $D' = \text{diag}(\nu_1 X_u, \dots, \nu_k X_u)$  for some  $u \in T$  and  $\nu_i \in \mathbb{F}^\times$  satisfying

$$(12) \quad \nu_1^2 \beta(u, t_1) = \dots = \nu_q^2 \beta(u, t_q) = \nu_{q+1} \nu_{q+2} = \dots = \nu_{q+2s-1} \nu_{q+2s}.$$

It follows at once that, for  $\psi \in \text{Aut}^*(\Gamma, \varphi)$ , the permutation  $\pi$  must preserve the set  $\{1, \dots, q\}$  and the pairing of  $q+2i-1$  with  $q+2i$ , for  $i = 1, \dots, s$ . It is convenient to introduce the group  $W(s) := \mathbb{Z}_2^s \rtimes \text{Sym}(s)$  (i.e., the wreath product of  $\text{Sym}(s)$  and  $\mathbb{Z}_2$ ), which will be regarded as the group of permutations on  $\{q+1, \dots, q+2s\}$  that respect the block decomposition  $\{q+1, q+2\} \cup \dots \cup \{q+2s-1, q+2s\}$ . The reason for the notation  $W(s)$  is that  $\mathbb{Z}_2^s \rtimes \text{Sym}(s)$  is the classical Weyl group of type  $B_s$  or  $C_s$  (and also the extended Weyl group of type  $D_s$  if  $s > 4$ ). By the above discussion, we have a homomorphism:

$$(13) \quad \text{Aut}^*(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times \text{Aut}(T, \beta).$$

We need some more notation to state the main result of this section. Let  $\Sigma$  be a multiset of cardinality  $q$  and let  $m_1, \dots, m_\ell$  be the multiplicities of the elements of  $\Sigma$ , written in some order. Thus,  $m_i$  are positive integers whose sum is  $q$ . We will denote by  $\text{Sym}\Sigma$  the subgroup  $\text{Sym}(m_1) \times \dots \times \text{Sym}(m_\ell)$  of  $\text{Sym}(q)$ , which may be thought of as “interior symmetries” of  $\Sigma$ . For a multiset  $\Sigma$  in  $T$ , let  $\text{Aut}^* \Sigma$  be the stabilizer of  $\Sigma$  under the natural action of  $T \rtimes \text{Aut}(T, \beta)$  on  $T$ , i.e.,  $\text{Aut}^* \Sigma$

is the set of “rigid motions” of the symplectic space  $T$  that permute the elements of  $\Sigma$  preserving multiplicity. These are “exterior symmetries” of  $\Sigma$ . Note that each bijection  $\theta: T \rightarrow T$  that stabilizes  $\Sigma$  determines an element of  $\text{Sym}(q)$  that permutes the blocks of sizes  $m_1, \dots, m_\ell$  in the same way  $\theta$  permutes the elements of  $\Sigma$  (thus, only blocks of equal size may be permuted) and preserves the order within each block; we will call this permutation the *restriction of  $\theta$  to  $\Sigma$* . Hence, we obtain a restriction homomorphism  $\text{Aut}^* \Sigma \rightarrow \text{Sym}(q)$ . In particular,  $\text{Aut}^* \Sigma$  acts naturally on  $\text{Sym} \Sigma$  by permuting factors (of equal order). Finally, let  $\text{Aut} \Sigma$  be the stabilizer of  $\Sigma$  under the twisted action of  $\text{Aut}(T, \beta)$  on  $T$  as in Definition 3.9. Note that  $\text{Aut} \Sigma$  may be regarded as a subgroup of  $\text{Aut}^* \Sigma$ .

**Theorem 3.12.** *Let  $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$  and let  $\varphi$  be as in Theorem 3.5 such that  $\Gamma$  is a fine  $\varphi$ -grading. Let  $\Sigma = \Sigma(\tau)$ , so  $|\Sigma| = q$ .*

- 1)  $\text{Stab}(\Gamma, \varphi) = \text{Diag}(\Gamma)$ .
- 2)  $\text{Aut}^*(\Gamma, \varphi) / \text{Stab}(\Gamma, \varphi)$  is isomorphic to an extension of the group  $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym} \Sigma \times \text{Sym}(s))) \rtimes \text{Aut}^* \Sigma$  by  $\mathbb{Z}_2^{q+s-1}$ , with the following actions:  $T^{q+s-1}$  is identified with  $T^{q+s}/T$  and  $\mathbb{Z}_2^{q+s-1}$  is identified with  $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ , where  $T$  and  $\mathbb{Z}_2$  are imbedded diagonally, then
  - $\text{Sym} \Sigma \subset \text{Sym}(q)$  acts on  $T^{q+s}/T$  and  $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$  by permuting the first  $q$  components and trivially on  $\mathbb{Z}_2^s$ ;
  - $\text{Sym}(s)$  acts on  $T^{q+s}/T$  and  $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$  by permuting the last  $s$  components and naturally on  $\mathbb{Z}_2^s$ ;
  - $\text{Aut}^* \Sigma$  acts on  $\text{Sym} \Sigma$  and  $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$  through the restriction homomorphism  $\text{Aut}^* \Sigma \rightarrow \text{Sym}(q)$ , trivially on  $\text{Sym}(s)$ , and as follows on  $(T^{q+s}/T) \times \mathbb{Z}_2^s$ : an element  $(u, \alpha) \in \text{Aut}^* \Sigma \subset T \rtimes \text{Aut}(T, \beta)$  sends a pair  $((u_1, \dots, u_q, u_{q+1}, \dots, u_{q+s})T, \underline{x}) \in (T^{q+s}/T) \times \mathbb{Z}_2^s$  to  $((\alpha(u_{\pi^{-1}(1)}), \dots, \alpha(u_{\pi^{-1}(q)}), \alpha(u_{q+1})u^{x_1}, \dots, \alpha(u_{q+s})u^{x_s})T, \underline{x})$ , where  $\pi$  is the image of  $(u, \alpha)$  under the restriction homomorphism;
  - $T^{q+s-1} \times \mathbb{Z}_2^s$  acts trivially on  $\mathbb{Z}_2^{q+s-1}$ .
- 3) If  $\varphi$  is an involution, then  $\text{Aut}(\Gamma, \varphi) / \text{Stab}(\Gamma, \varphi)$  is isomorphic to  $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym} \Sigma \times \text{Sym}(s))) \rtimes \text{Aut} \Sigma$ , with the following actions:  $T^{q+s-1}$  is identified with  $T^{q+s}/T$ , where  $T$  is imbedded diagonally, then
  - $\text{Sym} \Sigma \subset \text{Sym}(q)$  acts on  $T^{q+s}/T$  by permuting the first  $q$  components and trivially on  $\mathbb{Z}_2^s$ ;
  - $\text{Sym}(s)$  acts on  $T^{q+s}/T$  by permuting the last  $s$  components and naturally on  $\mathbb{Z}_2^s$ ;
  - $\text{Aut} \Sigma$  acts on  $\text{Sym} \Sigma$  as a subgroup of  $\text{Aut}^* \Sigma$ , i.e., through the twisted action on  $T$  (Definition 3.9) and restriction to  $\Sigma$ , trivially on  $\text{Sym}(s)$ , and as follows on  $(T^{q+s}/T) \times \mathbb{Z}_2^s$ : an element  $\alpha \in \text{Aut} \Sigma \subset \text{Aut}(T, \beta)$  sends a pair  $((u_1, \dots, u_q, u_{q+1}, \dots, u_{q+s})T, \underline{x}) \in (T^{q+s}/T) \times \mathbb{Z}_2^s$  to  $((\alpha(u_{\pi^{-1}(1)}), \dots, \alpha(u_{\pi^{-1}(q)}), \alpha(u_{q+1})t_\alpha^{x_1}, \dots, \alpha(u_{q+s})t_\alpha^{x_s})T, \underline{x})$ , where  $\pi$  is the image of  $(t_\alpha, \alpha)$  under the restriction to  $\Sigma$ .

*Proof.* 1) If  $\psi \in \text{Stab}(\Gamma, \varphi)$ , then  $\Psi = PD$  where  $P$  corresponds to  $\pi \in \text{Sym}(q) \times W(s)$ , and  $\psi_0 \in \text{Stab}(\Gamma_0)$ . Adjusting  $D$  if necessary, we may assume  $\psi_0 = \text{id}$ . We claim that  $\pi$  is the trivial permutation. Since  $\psi$  does not permute the homogeneous components of  $\Gamma$ ,  $\pi$  must act trivially on  $\tilde{G}^0/T$ . So, we consider the action of  $\text{Sym}(q) \times W(s)$  on  $\tilde{G}^0/T$  in terms of the generators  $z_i$  ( $i = 1, \dots, q-1$  if  $s = 0$  and  $i = 1, \dots, q+s$  if  $s > 0$ ) that were introduced after Definition 3.3.

$\text{Sym}(q)$  acts trivially on the subgroup  $\langle z_{q+1}, \dots, z_{q+s} \rangle$  and via the action of the classical Weyl group of type  $A_{q-1}$ , taken modulo 2, on the subgroup  $\langle z_1, \dots, z_{q-1} \rangle \cong \mathbb{Z}_2^{q-1}$  where  $z_i$  is identified with the element  $\varepsilon_i - \varepsilon_{i+1}$ , with  $\{\varepsilon_1, \dots, \varepsilon_q\}$  being the standard basis of  $\mathbb{Z}_2^q$ , on which  $\text{Sym}(q)$  acts naturally.

$W(s)$  acts trivially on the subgroup  $\langle z_1, \dots, z_{q-1} \rangle$  and via the action of the classical Weyl group of type  $B_s$  or  $C_s$  on the subgroup  $\langle z_{q+1}, \dots, z_{q+s} \rangle \cong \mathbb{Z}^s$  where  $z_{q+i}$  is identified with the element  $\varepsilon_i - \varepsilon_{i+1}$  for  $i \neq s$  and  $z_{q+s}$  is identified with the element  $2\varepsilon_1$ , with  $\{\varepsilon_1, \dots, \varepsilon_s\}$  being the standard basis of  $\mathbb{Z}^s$ . The easiest way to see this is to extend  $\tilde{G}$  by adding a new element  $\hat{g}_0$  satisfying  $(\hat{g}_0)^{-2} = \tilde{g}_1 \tilde{g}_2$  and set  $\hat{g}_i = \tilde{g}_i \hat{g}_0$ . The elements of the subgroup  $\tilde{G}^0$  are not affected if we replace  $\tilde{g}_i$  by  $\hat{g}_i$ , but then we have  $\hat{g}_{q+2j} = \hat{g}_{q+2j-1}^{-1}$  for  $j = 1, \dots, s$ , so we can map  $\hat{g}_{q+2j-1}$  to  $\varepsilon_j$  and  $\hat{g}_{q+2j}$  to  $-\varepsilon_j$ .

Note that the action of  $W(s)$  on  $\langle z_{q+1}, \dots, z_{q+s} \rangle$  is always faithful, while the action of  $\text{Sym}(q)$  on  $\langle z_1, \dots, z_{q-1} \rangle$  is faithful unless  $q = 2$ . If  $q > 0$  and  $s > 0$ , then we also have the generator  $z_q$ , on which  $\pi \in \text{Sym}(q) \times W(s)$  acts in this way (note that  $\pi(q) \leq q$  and  $\pi(q+1) > q$ ):

$$z_q \mapsto \begin{cases} z_{\pi(q)} \cdots z_q z_{q+1} \cdots z_{q+j} & \text{if } \pi(q+1) = q+2j+1; \\ z_{\pi(q)} \cdots z_q z_{q+1}^{-1} \cdots z_{q+j}^{-1} z_{q+s} & \text{if } \pi(q+1) = q+2j+2. \end{cases}$$

If  $\pi$  acts trivially on  $\langle z_{q+1}, \dots, z_{q+s} \rangle$ , then  $\pi(q+1) = q+1$ . Hence, if  $\pi$  also acts trivially on  $z_q$ , then  $\pi(q) = q$ . It follows that the action of  $\text{Sym}(q) \times W(s)$  on  $\tilde{G}^0/T$  is faithful unless  $q = 2$  and  $s = 0$ . In this remaining case, we have  $\tau = (t_1, t_2)$  where  $t_1 \neq t_2$  (otherwise  $\Gamma$  is not a fine  $\varphi$ -grading). If  $\psi_1$  yields  $\pi = (12)$ , then  $\psi_1(v_1) = v_2 d_1$  and  $\psi(v_2) = v_1 d_2$  for some nonzero homogeneous  $d_1, d_2 \in \mathcal{D}$ , but then  $B(\psi_1(v_1), \psi_1(v_1))$  has degree  $t_2$ , while  $B(v_1, v_1)$  has degree  $t_1$ . This contradicts (11), because here we have  $\psi_0 = \text{id}$ ,  $d_0 \in \mathbb{F}^\times$  and  $B' = B$ . The proof of the claim is complete.

Since  $P = I$ , we have  $\Psi = \text{diag}(d_1, \dots, d_k)$ , where the  $d_i$  must necessarily have the same degree, say,  $t$ , so  $\Psi = \text{diag}(\lambda_1, \dots, \lambda_k) \otimes X_t$ , but then (11) implies that (5) must hold, hence  $\psi \in \text{Diag}(\Gamma)$ . We have proved that  $\text{Stab}(\Gamma, \varphi) \subset \text{Diag}(\Gamma)$ . The opposite inclusion is obvious.

2) We can extract more information about an element  $\psi \in \text{Aut}^*(\Gamma, \varphi)$  than given by its image under the homomorphism (13) if we look at the action of  $\psi$  on  $\varphi$ . Write  $\psi\varphi\psi^{-1} = \xi_\psi\varphi$  where  $\xi_\psi$  is a uniquely determined element of  $\text{Diag}(\Gamma)$ . Clearly, we have  $\xi_{\psi\psi'} = \xi_\psi(\psi\xi_{\psi'}\psi^{-1})$ . Since  $\xi_\psi$  is the conjugation by  $\text{diag}(\nu_1, \dots, \nu_k) \otimes X_{u_\psi}$ , for a uniquely determined  $u_\psi \in T$ , we obtain  $u_{\psi\psi'} = u_\psi \alpha_\psi(u_{\psi'})$  where  $\alpha_\psi$  is the element of  $\text{Aut}(T, \beta)$  corresponding to  $\psi$  under (13). Hence, we can construct a homomorphism

$$(14) \quad \text{Aut}^*(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times (T \rtimes \text{Aut}(T, \beta)),$$

where the first two components are as in (13) and the third is  $\psi \mapsto (u_\psi, \alpha_\psi)$ .

Theorem 3.8 implies that we may assume without loss of generality that

$$\Phi = \text{diag} \left( X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right).$$

(In other words, the scalars  $\mu_i$  are all equal to 1.) Then, for  $\psi$  given by  $\Psi = PD$  and  $\psi_0 \in \text{Aut}(\Gamma_0)$ , with  $P$  corresponding to  $\pi \in \text{Sym}(q) \times W(s)$ , condition (11) is

equivalent to the following, with  $u = u_\psi$ :

$$(15) \quad \varphi_0(d_i)X_{t_{\pi(i)}}\nu_{\pi(i)}^{-1}X_u^{-1}d_i = d_0\psi_0(X_{t_i}), \quad i = 1, \dots, q,$$

and, for each  $j = 1, \dots, s$ , one of the following depending on whether  $\pi(q+2j-1) < \pi(q+2j)$  or  $\pi(q+2j-1) > \pi(q+2j)$ :

$$(16) \quad \varphi_0(d_{q+2j-1})\nu_{\pi(q+2j)}^{-1}X_u^{-1}d_{q+2j} = \varphi_0(d_{q+2j})\nu_{\pi(q+2j-1)}^{-1}X_u^{-1}d_{q+2j-1} = d_0$$

in the first case, and

$$(17) \quad \varphi_0(d_{q+2j-1})\nu_{\pi(q+2j-1)}^{-1}X_u^{-1}d_{q+2j} = \varphi_0(d_{q+2j})\nu_{\pi(q+2j)}^{-1}X_u^{-1}d_{q+2j-1} = d_0$$

in the second case.

If  $\psi \in \text{Aut}^*(\Gamma, \varphi)$ , then, looking at the degrees in (15), we obtain

$$(18) \quad t_{\pi(i)} = \alpha_\psi(t_i)t_{\alpha_\psi}u_\psi, \quad i = 1, \dots, q,$$

which implies that  $(t_{\alpha_\psi}u_\psi, \alpha_\psi)$  belongs to  $\text{Aut}^*\Sigma$ . Composing the third component of the homomorphism (14) with the automorphism  $(u, \alpha) \mapsto (t_\alpha u, \alpha)$  of the group  $T \rtimes \text{Aut}(T, \beta)$ , we obtain a homomorphism

$$(19) \quad \text{Aut}^*(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times \text{Aut}^*\Sigma.$$

For any element  $(t_\alpha u, \alpha) \in \text{Aut}^*\Sigma$ , let  $\pi_{u, \alpha} \in \text{Sym}(q)$  be its restriction to  $\Sigma$ . Then (18) implies that the permutation  $\pi\pi_{u, \alpha}^{-1}$  does not move the elements of the underlying set of  $\Sigma$ , so it belongs to  $\text{Sym}\Sigma$ . It follows that (19) can be rearranged as follows:

$$f: \text{Aut}^*(\Gamma, \varphi) \rightarrow W(s) \times (\text{Sym}\Sigma \rtimes \text{Aut}^*\Sigma).$$

We claim that  $f$  is surjective. We will construct representatives in  $\text{Aut}^*(\Gamma, \varphi)$  for the elements of each of the subgroups  $W(s)$ ,  $\text{Sym}\Sigma$  and  $\text{Aut}^*\Sigma$ .

For any  $\pi \in W(s)$ , let  $P$  be the corresponding permutation matrix and let  $\psi_\pi$  be given by  $\Psi = P$  and  $\psi_0 = \text{id}$ . Let  $\alpha$  be the automorphism of  $\tilde{G}$  that restricts to identity on  $T$  and sends  $\tilde{g}_i$  to  $\tilde{g}_{\pi(i)}$  (in particular,  $\tilde{g}_i$  are fixed for  $i = 1, \dots, q$ ). Then  $\psi_\pi$  sends  ${}^\alpha\Gamma$  to  $\Gamma$ , so  $\psi_\pi \in \text{Aut}(\Gamma)$ . Also, conditions (15) through (17) are satisfied with  $d_0 = I$ ,  $u = e$  and  $\nu_i = 1$ , so  $\psi_\pi \in \text{Aut}(\Gamma, \varphi)$ .

For any  $\pi \in \text{Sym}\Sigma$ , let  $P$  be the corresponding permutation matrix and let  $\psi_\pi$  be given by  $\Psi = P$  and  $\psi_0 = \text{id}$ . Since we have  $t_{\pi(i)} = t_i$  for all  $i = 1, \dots, q$ , we can define the automorphism  $\alpha$  of  $\tilde{G}$  in the same way as above (this time,  $\tilde{g}_i$  are fixed for  $i = q+1, \dots, q+2s$ ). Then  $\psi_\pi$  sends  ${}^\alpha\Gamma$  to  $\Gamma$ , so  $\psi_\pi \in \text{Aut}(\Gamma)$ . Also, conditions (15) and (16) are satisfied with  $d_0 = I$ ,  $u = e$  and  $\nu_i = 1$ , so  $\psi_\pi \in \text{Aut}(\Gamma, \varphi)$ .

Now, for any  $(t_\alpha u, \alpha) \in \text{Aut}^*\Sigma$ , let  $\pi = \pi_{u, \alpha}$ . Then  $t_{\pi(i)} = \alpha(t_i)t_\alpha u$  for  $i = 1, \dots, q$  and hence we can extend  $\alpha: T \rightarrow T$  to an automorphism of  $\tilde{G}$  by setting  $\alpha(\tilde{g}_i) = \tilde{g}_{\pi(i)}$  for  $i = 1, \dots, q$ ,  $\alpha(\tilde{g}_{q+2j-1}) = \tilde{g}_{q+2j-1}$  and  $\alpha(\tilde{g}_{q+2j}) = \tilde{g}_{q+2j}t_\alpha u$  for  $j = 1, \dots, s$ . Choose  $\nu_i \in \mathbb{F}^\times$  such that  $\nu_i^2 = \beta(u, t_i)\beta(u)$ ,  $i = 1, \dots, q$ , and set  $\nu_{q+2j} = 1$  and  $\nu_{q+2j-1} = \beta(u)$ ,  $j = 1, \dots, s$ . Then (12) holds, so the conjugation by  $\text{diag}(\nu_1 X_u, \dots, \nu_k X_u)$  is an element  $\xi \in \text{Diag}(\Gamma)$ . Choose  $\psi_0$  such that  $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$ . Let  $P$  be the permutation matrix corresponding to  $\pi$  and let

$$D = \text{diag}(\lambda_1 I, \dots, \lambda_q I, I, X_u X_{t_\alpha}, \dots, I, X_u X_{t_\alpha}),$$

where  $\lambda_i \in \mathbb{F}^\times$  are selected in such a way that condition (15) holds with  $d_0 = X_{t_\alpha}$  (the degrees of both sides match, so it is indeed possible to find such  $\lambda_i$ ). Since  $\beta(t_\alpha) = 1$ , condition (16) also holds. Finally, let  $\psi_{u, \alpha}$  be given by  $\Psi = PD$  and  $\psi_0$ .

Then  $\psi_{u,\alpha}$  sends  ${}^\alpha\Gamma$  to  $\Gamma$  and  $\varphi$  to  $\xi\varphi$ , with  $\alpha$  and  $\xi$  indicated above. Therefore,  $\psi_{u,\alpha}$  belongs to  $\text{Aut}^*(\Gamma, \varphi)$ .

We have proved that the homomorphism  $f$  is surjective. Let  $K$  be the kernel of  $f$ . It consists of the conjugations by matrices of the form  $D = \text{diag}(d_1, \dots, d_k)$  such that (15) and (16) are satisfied with  $\pi = \text{id}$ ,  $\psi_0 = \text{id}$ ,  $d_0 \in \mathbb{F}^\times$  and  $u = e$ . Hence  $\deg d_{q+2j-1} = \deg d_{q+2j}$  for all  $j = 1, \dots, s$ . Conversely, given  $(u_1, \dots, u_k) \in T^k$  with  $u_{q+2j-1} = u_{q+2j}$  for  $j = 1, \dots, s$ , we can find elements  $d_i$  with  $\deg d_i = u_i$  such that the conjugation by  $D$  belongs to  $\text{Aut}(\Gamma, \varphi)$ .

According to 1), the subgroup

$$N = \{\psi \in K \mid \deg d_1 = \dots = \deg d_k\}$$

contains  $\text{Stab}(\Gamma, \varphi)$ . Clearly,  $N$  is normal in  $\text{Aut}^*(\Gamma, \varphi)$ . From the previous paragraph it follows that  $K/N \cong T^{q+s}/T$  where  $T$  is imbedded into  $T^{q+s}$  diagonally. The representatives  $\psi_\pi$  that we constructed above for  $\pi \in W(s)$  and for  $\pi \in \text{Sym}\Sigma$  form subgroups of  $\text{Aut}(\Gamma, \varphi)$  that commute with one another. But observe also that the representatives  $\psi_{u,\alpha}$  for  $(t_\alpha u, \alpha) \in \text{Aut}^*\Sigma$  form a subgroup modulo  $N$ . Moreover, for  $\pi \in \text{Sym}(s) \subset W(s)$  the elements  $\psi_{u,\alpha}$  and  $\psi_\pi$  commute modulo  $N$ , while for  $\pi \in \text{Sym}\Sigma$  we have  $\psi_{u,\alpha}\psi_\pi\psi_{u,\alpha}^{-1} \in \psi_{\pi u, \alpha\pi\pi_{u,\alpha}^{-1}}N$ . Finally, for the transposition  $\pi = (q+2j-1, q+2j)$ , we have  $\psi_\pi\psi_{u,\alpha}\psi_\pi\psi_{u,\alpha}^{-1} \in \psi N$  where  $\psi$  is the conjugation by  $\text{diag}(d_1, \dots, d_k)$  with  $d_{q+2j-1} = d_{q+2j} = X_{t_\alpha u}$  and all other  $d_i = I$ . It follows that  $\text{Aut}^*(\Gamma, \varphi)/N$  is isomorphic to  $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym}\Sigma \times \text{Sym}(s))) \rtimes \text{Aut}^*\Sigma$ , with the stated actions.

It remains to compute the quotient  $N/\text{Stab}(\Gamma, \varphi)$ . Since any element  $\psi \in N$  belongs to  $\text{Stab}(\Gamma)$ , the mapping  $\psi \mapsto \xi_\psi$  is a homomorphism  $N \rightarrow \text{Diag}(\Gamma)$  whose kernel is exactly  $\text{Stab}(\Gamma, \varphi)$ . Hence, it suffices to compute the image. Since here  $u = e$  and  $\deg d_{q+2j-1} = \deg d_{q+2j}$ , condition (16) implies that  $\nu_{q+2j-1} = \nu_{q+2j}$  for  $j = 1, \dots, s$ . But then (12) implies that all  $\nu_i^2$  are equal to each other. Since multiplying all  $\nu_i$  by the same scalar in  $\mathbb{F}^\times$  does not change  $\xi$ , we may assume that  $\nu_i \in \{\pm 1\}$ . In fact, for  $D = \text{diag}(\lambda_1 I, \dots, \lambda_k I)$ , conditions (15) and (16) reduce to the following: up to a common scalar multiple,  $\nu_i = \lambda_i^2$  for  $i = 1, \dots, q$ , and  $\nu_{q+2j-1} = \nu_{q+2j} = \lambda_{q+2j-1}\lambda_{q+2j}$  for  $j = 1, \dots, s$ . Hence every  $(\nu_1, \dots, \nu_k)$  with  $\nu_i \in \{\pm 1\}$  and  $\nu_{q+2j-1} = \nu_{q+2j}$  indeed appears in  $\xi_\psi$  for some  $\psi \in N$ . Therefore, the quotient  $N/\text{Stab}(\Gamma, \varphi)$  is isomorphic to  $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$  where  $\mathbb{Z}_2$  is imbedded into  $\mathbb{Z}_2^{q+s}$  diagonally.

3) The proof is similar to 2), so we will merely point out the differences. According to Theorem 3.10, here we have

$$\Phi = \text{diag} \left( X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix} \right),$$

where  $\delta = \text{sgn}(\varphi)$  and  $\beta(t_i) = \delta$  for  $i = 1, \dots, q$ . Also,  $B'$  equals  $B$  and hence, for  $\psi$  given by  $\Psi = PD$  and  $\psi_0 \in \text{Aut}(\Gamma_0)$ , with  $P$  corresponding to  $\pi \in \text{Sym}(q) \times W(s)$ , condition (11) is equivalent to the following:

$$(20) \quad \varphi_0(d_i)X_{t_{\pi(i)}}d_i = d_0\psi_0(X_{t_i}), \quad i = 1, \dots, q,$$

and, for each  $j = 1, \dots, s$ , one of the following depending on whether  $\pi(q+2j-1) < \pi(q+2j)$  or  $\pi(q+2j-1) > \pi(q+2j)$ :

$$(21) \quad \varphi_0(d_{q+2j-1})d_{q+2j} = d_0$$

in the first case, and

$$(22) \quad \varphi_0(d_{q+2j-1})d_{q+2j} = \delta d_0$$

in the second case. Here we took into account that, since  $\varphi_0(d_0) = d_0$ , either (21) or (22) implies  $\varphi_0(d_{q+2j-1})d_{q+2j} = \varphi_0(d_{q+2j})d_{q+2j-1}$ .

If  $\psi \in \text{Aut}(\Gamma, \varphi)$ , then, looking at the degrees in (20), we obtain

$$(23) \quad t_{\pi(i)} = \alpha_\psi(t_i)t_{\alpha_\psi}, \quad i = 1, \dots, q,$$

which implies that  $(t_{\alpha_\psi}, \alpha_\psi)$  stabilizes  $\Sigma$ , i.e.,  $\alpha_\psi$  belongs to  $\text{Aut } \Sigma$ . Hence we obtain a homomorphism

$$(24) \quad \text{Aut}(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times \text{Aut } \Sigma.$$

For any element  $\alpha \in \text{Aut } \Sigma$ , let  $\pi_\alpha \in \text{Sym}(q)$  be the restriction of its twisted action to  $\Sigma$ . Then (23) implies that the permutation  $\pi\pi_{\alpha_\psi}^{-1}$  does not move the elements of the underlying set of  $\Sigma$ , so it belongs to  $\text{Sym } \Sigma$ . It follows that (24) can be rearranged as follows:

$$f: \text{Aut}(\Gamma, \varphi) \rightarrow W(s) \times (\text{Sym } \Sigma \rtimes \text{Aut } \Sigma).$$

To prove that  $f$  is surjective, we construct representatives in  $\text{Aut}(\Gamma, \varphi)$  for the elements of each of the subgroups  $W(s)$ ,  $\text{Sym } \Sigma$  and  $\text{Aut } \Sigma$ .

For  $\pi$  in  $\text{Sym } \Sigma$  or in  $\text{Sym}(s) \subset W(s)$ , we take the same representatives as in the proof of 2). For  $\pi = (q+2j-1, q+2j) \in W(s)$ , a slight modification is needed: we take  $\Psi = PD$  rather than just  $P$ , where  $d_{q+2j} = \delta I$  and all other  $d_i = I$ . For any  $\alpha \in \text{Aut } \Sigma$ , let  $\pi = \pi_\alpha$ . Then  $t_{\pi(i)} = \alpha(t_i)t_\alpha$  for  $i = 1, \dots, q$  and hence we can extend  $\alpha: T \rightarrow T$  to an automorphism of  $\tilde{G}$  by setting  $\alpha(\tilde{g}_i) = \tilde{g}_{\pi(i)}$  for  $i = 1, \dots, q$ ,  $\alpha(\tilde{g}_{q+2j-1}) = \tilde{g}_{q+2j-1}$  and  $\alpha(\tilde{g}_{q+2j}) = \tilde{g}_{q+2j}t_\alpha$  for  $j = 1, \dots, s$ . Choose  $\psi_0$  such that  $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$ . Let  $P$  be the permutation matrix corresponding to  $\pi$  and let

$$D = \text{diag}(\lambda_1 I, \dots, \lambda_q I, I, X_{t_\alpha}, \dots, I, X_{t_\alpha}),$$

where  $\lambda_i \in \mathbb{F}^\times$  are selected in such a way that condition (20) holds with  $d_0 = X_{t_\alpha}$ . Clearly, condition (21) also holds. Finally, let  $\psi_\alpha$  be given by  $\Psi = PD$  and  $\psi_0$ . Then  $\psi_\alpha$  sends  ${}^\alpha\Gamma$  to  $\Gamma$  and fixes  $\varphi$ , so  $\psi_\alpha$  belongs to  $\text{Aut}(\Gamma, \varphi)$ .

Let  $K$  be the kernel of  $f$  and let

$$N = \{\psi \in K \mid \deg d_1 = \dots = \deg d_k\}.$$

The same arguments as in 2) show that  $K/N \cong T^{q+s}/T$  and  $\text{Aut}(\Gamma, \varphi)/N$  is isomorphic to  $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym } \Sigma \times \text{Sym}(s))) \rtimes \text{Aut } \Sigma$ , with the stated actions. But here we have  $N = \text{Stab}(\Gamma, \varphi)$ , which completes the proof.  $\square$

#### 4. SERIES A

In this section we describe the Weyl groups of fine gradings on the simple Lie algebras of series A. Thus, we take  $\mathcal{R} = M_n(\mathbb{F})$ ,  $n \geq 2$ , and  $\mathcal{L} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{R}, \mathcal{R}]/(Z(\mathcal{R}) \cap [\mathcal{R}, \mathcal{R}])$ . First we review the classification of fine gradings on  $\mathcal{L}$  from [Eld10] (extended to positive characteristic using automorphism group schemes) and then derive the Weyl groups for  $\mathcal{L}$  from what we already know about automorphisms of fine gradings ([EKb]) and fine  $\varphi$ -gradings (Section 3) on  $\mathcal{R}$ .

**4.1. Classification of fine gradings.** The case  $n = 2$  is easy, because the restriction from  $\mathcal{R}$  to  $\mathcal{L}$  yields an isomorphism  $\mathbf{Aut}(\mathcal{R}) \rightarrow \mathbf{Aut}(\mathcal{L})$ . It follows that the classification of fine gradings on  $\mathcal{L}$  is the same as that on  $\mathcal{R}$ . Namely, there are two fine gradings on  $\mathfrak{sl}_2(\mathbb{F})$ , up to equivalence: the Cartan grading, whose universal group is  $\mathbb{Z}$ , and the Pauli grading, whose universal group is  $\mathbb{Z}_2^2$ .

Now assume  $n \geq 3$ . Then the restriction and passing modulo the center yields a closed imbedding  $\mathbf{Aut}(\mathcal{R}) \rightarrow \mathbf{Aut}(\mathcal{L})$ , which is not an isomorphism. To rectify this, one introduces the affine group scheme  $\overline{\mathbf{Aut}}(\mathcal{R})$  corresponding to the algebraic group of automorphisms and anti-automorphisms of  $\mathcal{R}$  (see [BK10, §3]). Unless  $n = \text{char } \mathbb{F} = 3$ , we obtain an isomorphism  $\overline{\mathbf{Aut}}(\mathcal{R}) \rightarrow \mathbf{Aut}(\mathcal{L})$ . It is convenient to divide gradings on  $\mathcal{L}$  into two types: for Type I the corresponding diagonalizable subgroup scheme of  $\mathbf{Aut}(\mathcal{L})$  is contained in the image of the closed imbedding  $\mathbf{Aut}(\mathcal{R}) \rightarrow \mathbf{Aut}(\mathcal{L})$ , while for Type II it is not. In other words, a grading on  $\mathcal{L}$  is of Type I if and only if it is induced from a (unique) grading on  $\mathcal{R}$  by restriction and passing modulo the center.

In [BK10], the *distinguished element* of a Type II grading  $\Gamma$  is introduced. It can be characterized as the unique element  $h$  of order 2 in the grading group  $G$  such that the coarsening  $\overline{\Gamma}$  induced from  $\Gamma$  by the quotient map  $G \rightarrow \overline{G} := G/\langle h \rangle$  is a Type I grading. The original grading  $\Gamma$  can be recovered from  $\overline{\Gamma}$  if we know the action of some character  $\chi$  of  $G$  with  $\chi(h) = -1$ . Indeed, we just have to split each component of  $\overline{\Gamma}$  into eigenspaces with respect to the action of  $\chi$ . We can transfer this procedure to  $\mathcal{R}$  in the following way. The action of  $\chi$  on  $\mathcal{L}$  is induced by  $-\varphi$  where  $\varphi$  is an anti-automorphism of  $\mathcal{R}$ . The Type I grading  $\overline{\Gamma}$  on  $\mathcal{L}$  comes from a grading  $\overline{\Gamma}'$  on  $\mathcal{R}$ . Since  $-\varphi$  is an automorphism of  $\mathcal{R}^{(-)}$  (the Lie algebra  $\mathcal{R}$  under commutator) and  $\varphi^2$  acts as a scalar on each component of  $\overline{\Gamma}'$ , we can refine the  $\overline{G}$ -grading  $\overline{\Gamma}' : \mathcal{R} = \bigoplus_{\overline{g} \in \overline{G}} \mathcal{R}_{\overline{g}}$  to a  $G$ -grading  $\Gamma' : \mathcal{R}^{(-)} = \bigoplus_{g \in G} \mathcal{R}_g$  by splitting each component  $\mathcal{R}_{\overline{g}}$  into eigenspaces of  $\varphi$ . In detail,  $\varphi^2$  acts on  $\mathcal{R}_{\overline{g}}$  as multiplication by  $\chi^2(\overline{g})$  (where we regard  $\chi^2$  as a character of  $\overline{G}$ , since  $\chi^2(h) = 1$ ), so we set

$$(25) \quad \mathcal{R}_g = \{X \in \mathcal{R}_{\overline{g}} \mid \varphi(X) = -\chi(g)X\} = \{\varphi(X) - \chi(g)X \mid X \in \mathcal{R}_{\overline{g}}\}.$$

Then  $\Gamma'$  induces the original Type II grading  $\Gamma$  on  $\mathcal{L}$  by restriction and passing modulo the center.

Now we apply the above to fine gradings on  $\mathcal{L}$ . The fine gradings of Type I come from the fine gradings on  $\mathcal{R}$  that do not admit an anti-automorphism  $\varphi$  making them  $\varphi$ -gradings. All fine gradings on  $\mathcal{R}$  are obtained as follows. We start from  $T$ , a finite abelian group that admits a nondegenerate alternating bicharacter  $\beta$  (hence  $|T|$  is a square). Fix a realization,  $\mathcal{D}$ , of the matrix algebra endowed with a division grading with support  $T$  and bicharacter  $\beta$ . Let  $k \geq 1$  be an integer. Denote by  $\tilde{G} = \tilde{G}(T, k)$  the abelian group freely generated by  $T$  and the symbols  $\tilde{g}_1, \dots, \tilde{g}_k$ .

**Definition 4.1.** Let  $\mathcal{M}(\mathcal{D}, k)$  be the  $\tilde{G}$ -graded algebra  $\text{End}_{\mathcal{D}}(V)$  where  $V$  has a  $\mathcal{D}$ -basis  $\{v_1, \dots, v_k\}$  with  $\deg v_i = \tilde{g}_i$ . Let  $n = k\sqrt{|T|}$  and  $\mathcal{R} = M_n(\mathbb{F})$ . The grading on  $\mathcal{R}$  obtained by identifying  $\mathcal{R}$  with  $\mathcal{M}(\mathcal{D}, k)$  will be denoted by  $\Gamma_{\mathcal{M}}(\mathcal{D}, k)$ . In other words, we define this grading by identifying  $\mathcal{R} = M_k(\mathcal{D})$  and setting  $\deg(E_{ij} \otimes X_t) := \tilde{g}_i t \tilde{g}_j^{-1}$ . By abuse of notation, we will also write  $\Gamma_{\mathcal{M}}(T, k)$ .

The universal group of  $\Gamma_{\mathcal{M}}(T, k)$  is the subgroup  $\tilde{G}^0 = \tilde{G}(T, k)^0$  of  $\tilde{G}$  generated by the support, i.e., by the elements  $z_{i,j,t} := \tilde{g}_i t \tilde{g}_j^{-1}$ ,  $t \in T$ . Clearly,  $\tilde{G}^0 \cong T \times \mathbb{Z}^{k-1}$ . By [Eld10, Proposition 3.24],  $\Gamma_{\mathcal{M}}(T, k)$  is a  $\varphi$ -grading for some  $\varphi$  if and only if  $T$

is an elementary 2-group and  $k \leq 2$ . Two gradings,  $\Gamma_{\mathcal{M}}(T, k)$  and  $\Gamma_{\mathcal{M}}(T', k')$ , are equivalent if and only if  $T \cong T'$  and  $k = k'$ .

**Definition 4.2.** Consider the grading  $\Gamma_{\mathcal{M}}(T, k)$  on  $\mathcal{R}$  by the group  $\tilde{G}(T, k)^0$  where  $k \geq 3$  if  $T$  is an elementary 2-group. The  $\tilde{G}(T, k)^0$ -grading on  $\mathcal{L}$  obtained by restriction and passing modulo the center will be denoted by  $\Gamma_A^{(I)}(T, k)$ .

The grading  $\Gamma_A^{(I)}(T, k)$  is fine, and  $\tilde{G}(T, k)^0$  is its universal group. To deal with fine gradings of Type II, we will need the following general observation:

**Lemma 4.3.** *Let  $\bar{\Gamma}$  be a  $\varphi$ -grading on an algebra  $\mathcal{A}$  and let  $\bar{G}$  be its universal group. Then there exist an abelian group  $G$ , an element  $h \in G$  of order 2, a character  $\chi$  of  $G$  with  $\chi(h) = -1$  such that  $\bar{G} = G/\langle h \rangle$  and the action of  $\chi^2$  on the  $\bar{G}$ -graded algebra  $\mathcal{A}$  (regarding  $\chi^2$  as a character of the group  $\bar{G}$ ) coincides with  $\varphi^2$ . The pair  $(G, h)$  is determined uniquely up to isomorphism over  $\bar{G}$  (i.e.,  $\langle h \rangle \rightarrow G \rightarrow \bar{G}$  is unique up to equivalence of extensions).*

*Proof.* For each  $\bar{g} \in \bar{G}$ ,  $\varphi^2$  acts on  $\mathcal{A}_{\bar{g}}$  as multiplication by some  $\lambda(\bar{g}) \in \mathbb{F}^\times$ . Since  $\bar{G}$  is the universal group of  $\bar{\Gamma}$ ,  $\lambda: \bar{G} \rightarrow \mathbb{F}^\times$  is a homomorphism. For each  $\bar{g} \in \bar{G}$ , we select  $\mu(\bar{g}) \in \mathbb{F}^\times$  such that  $\mu(\bar{g})^2 = \lambda(\bar{g})$  (there are two choices). It will be convenient to choose  $\mu(\bar{e}) = 1$ . It follows that

$$(26) \quad \mu(\bar{x}\bar{y}) = \varepsilon(\bar{x}, \bar{y})\mu(\bar{x})\mu(\bar{y}) \quad \text{for all } \bar{x}, \bar{y} \in \bar{G}$$

where  $\varepsilon(\bar{x}, \bar{y}) \in \{\pm 1\}$ . One immediately verifies that  $\varepsilon$  is a symmetric 2-cocycle on  $\bar{G}$  with  $\varepsilon(\bar{g}, \bar{e}) = 1$  for all  $\bar{g} \in \bar{G}$  and, moreover, the class of  $\varepsilon$  in  $H^2(\bar{G}, \mathbb{Z}_2)$  (where we identified  $\{\pm 1\}$  with  $\mathbb{Z}_2$ ) does not depend on the choices of  $\mu(\bar{g})$ . Let  $G$  be the central extension of  $\bar{G}$  by  $\mathbb{Z}_2$  determined by  $\varepsilon$ , i.e.,  $G$  consists of the pairs  $(\bar{g}, \delta)$ ,  $\bar{g} \in \bar{G}$ ,  $\delta \in \{\pm 1\}$ , with multiplication given by

$$(27) \quad (\bar{x}, \delta_1)(\bar{y}, \delta_2) = (\bar{x}\bar{y}, \varepsilon(\bar{x}, \bar{y})\delta_1\delta_2) \quad \text{for all } \bar{x}, \bar{y} \in \bar{G} \text{ and } \delta_1, \delta_2 \in \{\pm 1\}.$$

Define  $\chi: G \rightarrow \mathbb{F}^\times$  by  $(\bar{g}, \delta) \mapsto \mu(\bar{g})\delta$ . Comparing (26) and (27), we see that  $\chi$  is a homomorphism. Set  $h = (\bar{e}, -1) \in G$ . Then  $h$  has order 2 and  $\chi(h) = -1$ . By construction, the action of  $\chi^2$  on  $\mathcal{A}$  determined by  $\bar{\Gamma}$  coincides with  $\varphi^2$ .  $\square$

Let  $T$  be an elementary 2-group of even dimension. Recall the group  $\tilde{G}(T, q, s, \tau)$ , which was introduced before Definition 3.3, and its subgroup  $\tilde{G}(T, q, s, \tau)^0$ .

**Definition 4.4.** Consider the grading  $\bar{\Gamma} = \Gamma_{\mathcal{M}}(T, q, s, \tau)$  on  $\mathcal{R}$  by the group  $\bar{G} = \tilde{G}(T, q, s, \tau)^0$  where  $t_1 \neq t_2$  if  $q = 2$  and  $s = 0$ . Let  $\Phi$  be the matrix given by

$$\Phi = \text{diag} \left( X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right).$$

Define  $\varphi(X) = \Phi^{-1}({}^t X)\Phi$ . Let  $G$ ,  $h$  and  $\chi$  be as in Lemma 4.3, so we obtain a  $G$ -grading on  $\mathcal{R}^{(-)}$  defined by (25). The  $G$ -grading on  $\mathcal{L}$  obtained by restriction and passing modulo the center will be denoted by  $\Gamma_A^{(II)}(T, q, s, \tau)$ .

The grading  $\Gamma_A^{(II)}(T, q, s, \tau)$  is fine, and  $G$  is its universal group. Note that  $\varphi^4 = \text{id}$ . It can be shown (cf. [Eld10, Example 3.21]) that the extension  $\langle h \rangle \rightarrow G \rightarrow \bar{G}$  is split if and only if there exists  $t \in T$  such that  $t_i t$  are in  $T_+$  for all  $i$  or in  $T_-$  for

all  $i$ . Taking into account (3), we see that  $G$  is isomorphic to

$$\begin{cases} \mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0, q-1) + 1} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s & \text{if } \exists t \in T \quad \beta(t_1 t) = \dots = \beta(t_q t); \\ \mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0, q-1)} \times \mathbb{Z}_4^{\dim T_0 + 1} \times \mathbb{Z}^s & \text{otherwise,} \end{cases}$$

where  $T_0$  is the subgroup of  $T$  generated by the elements  $t_i t_{i+1}$ ,  $i = 1, \dots, q-1$ .

Now Theorem 4.2 of [Eld10] can be extended to positive characteristic and recast as follows:

**Theorem 4.5.** *Let  $\mathbb{F}$  be an algebraically closed field,  $\text{char } \mathbb{F} \neq 2$ . Let  $n \geq 3$  if  $\text{char } \mathbb{F} \neq 3$  and  $n \geq 4$  if  $\text{char } \mathbb{F} = 3$ . Then any fine grading on  $\mathfrak{psl}_n(\mathbb{F})$  is equivalent to one of the following:*

- $\Gamma_A^{(I)}(T, k)$  as in Definition 4.2 with  $k\sqrt{|T|} = n$ ,
- $\Gamma_A^{(II)}(T, q, s, \tau)$  as in Definition 4.4 with  $(q + 2s)\sqrt{|T|} = n$ .

*Gradings belonging to different types listed above are not equivalent. Within each type, we have the following:*

- $\Gamma_A^{(I)}(T_1, k_1)$  and  $\Gamma_A^{(I)}(T_2, k_2)$  are equivalent if and only if  $T_1 \cong T_2$  and  $k_1 = k_2$ ;
- $\Gamma_A^{(II)}(T_1, q_1, s_1, \tau_1)$  and  $\Gamma_A^{(II)}(T_2, q_2, s_2, \tau_2)$  are equivalent if and only if  $T_1 \cong T_2$ ,  $q_1 = q_2$ ,  $s_1 = s_2$  and, identifying  $T_1 = T_2 = \mathbb{Z}_2^{2r}$ ,  $\Sigma(\tau_1)$  is conjugate to  $\Sigma(\tau_2)$  by the natural action of  $\text{ASp}_{2r}(2)$ .  $\square$

The missing case  $n = \text{char } \mathbb{F} = 3$  can be treated using octonions, because in characteristic 3 the algebra of traceless octonions under commutator is a Lie algebra isomorphic to  $\mathfrak{psl}_3(\mathbb{F})$  (cf. [BK10, Remark 4.11]).

**4.2. Weyl groups of fine gradings.** By [EKb, Theorem 2.8], the Weyl group of  $\Gamma_{\mathcal{M}}(T, k)$  is isomorphic to  $T^{k-1} \rtimes (\text{Sym}(k) \times \text{Aut}(T, \beta))$ , with  $\text{Sym}(k)$  and  $\text{Aut}(T, \beta)$  acting on  $T^{k-1}$  through their natural action on  $T^k$  and identification of  $T^{k-1}$  with  $T^k/T$  where  $T$  is imbedded into  $T^k$  diagonally. Thanks to the isomorphism  $\mathbf{Aut}(M_2(\mathbb{F})) \rightarrow \mathbf{Aut}(\mathfrak{sl}_2(\mathbb{F}))$ , it follows that the Weyl group of the Cartan grading on  $\mathfrak{sl}_2(\mathbb{F})$  is  $\text{Sym}(2)$  (the classical Weyl group of type  $A_1$ ) and the Weyl group of the Pauli grading on  $\mathfrak{sl}_2(\mathbb{F})$  is  $\text{Sp}_2(2) = \text{GL}_2(2)$  (this is known in the case  $\text{char } \mathbb{F} = 0$  — see [HPPT02]).

To state our result for  $\mathfrak{psl}_n(\mathbb{F})$ ,  $n \geq 3$ , it is convenient to introduce the following notation:

$$\overline{\text{Aut}}(T, \beta) := \text{Aut}(T, \beta) \rtimes \langle \sigma \rangle,$$

where  $\sigma$  is an element of order 2 acting as the automorphism of  $T$  that sends  $a_i$  to  $a_i^{-1}$  and  $b_i$  to  $b_i$ , where  $a_i$  and  $b_i$  are the generators of  $T$  used for the chosen realization of  $\mathcal{D}$  (a “symplectic basis” of  $T$  with respect to  $\beta$ ). We observe that  $\beta(\sigma \cdot u, \sigma \cdot v) = \beta(u, v)^{-1}$ , for all  $u, v \in T$ , and hence we obtain an induced action of  $\sigma$  on  $\text{Aut}(T, \beta)$  by setting  $(\sigma \cdot \alpha)(t) := \sigma \cdot \alpha(\sigma \cdot t)$  for all  $\alpha \in \text{Aut}(T, \beta)$  and  $t \in T$ . The elements of  $\overline{\text{Aut}}(T, \beta)$  act as automorphisms of  $T$  that send  $\beta$  to  $\beta^{\pm 1}$ . However, this action is not faithful if  $T$  is an elementary 2-group.

**Theorem 4.6.** *Let  $\mathbb{F}$  be an algebraically closed field,  $\text{char } \mathbb{F} \neq 2$ . Let  $n \geq 3$  if  $\text{char } \mathbb{F} \neq 3$  and  $n \geq 4$  if  $\text{char } \mathbb{F} = 3$ . Consider the fine grading  $\Gamma = \Gamma_A^{(I)}(T, k)$  on  $\mathfrak{psl}_n(\mathbb{F})$  as in Definition 4.2,  $k\sqrt{|T|} = n$ . Then*

$$W(\Gamma) \cong T^{k-1} \rtimes (\text{Sym}(k) \times \overline{\text{Aut}}(T, \beta)),$$

with  $\text{Sym}(k)$  and  $\overline{\text{Aut}}(T, \beta)$  acting on  $T^{k-1}$  through their natural action on  $T^k$  and identification of  $T^{k-1}$  with  $T^k/T$  where  $T$  is imbedded into  $T^k$  diagonally.

*Proof.* The grading  $\Gamma$  on  $\mathcal{L} = \mathfrak{psl}_n(\mathbb{F})$  is induced by the grading  $\Gamma' = \Gamma_{\mathcal{M}}(T, k)$  on  $\mathcal{R} = M_n(\mathbb{F})$ . The universal group of both gradings is  $G = \tilde{G}(T, k)^0$ . Since restriction is a bijection between gradings on  $\mathcal{R}$  and Type I gradings on  $\mathcal{L}$ , an automorphism  $\psi'$  of  $\mathcal{R}$  sends  ${}^\alpha\Gamma'$  to  $\Gamma'$ , for some automorphism  $\alpha$  of  $G$ , if and only if the induced automorphism  $\psi$  of  $\mathcal{L}$  sends  ${}^\alpha\Gamma$  to  $\Gamma$ . The automorphism group of  $\mathcal{L}$  is the semidirect product of  $\text{Aut}(\mathcal{R})$ , in its induced action on  $\mathcal{L}$ , and  $\langle \sigma \rangle$ , where  $\sigma$  is given by the negative of matrix transpose. To compute the action of  $\sigma$ , recall that  $(u_1, \dots, u_k)T \in T^k/T$  can be represented by the automorphism  $X \mapsto DXD^{-1}$  where  $D = \text{diag}(X_{u_1}, \dots, X_{u_k})$ ,  $\pi \in \text{Sym}(k)$  can be represented by  $X \mapsto PXP^{-1}$  where  $P$  is the permutation matrix corresponding to  $\pi$ , and  $\alpha \in \text{Aut}(T, \beta)$  can be represented by  $X \mapsto \psi_0(X)$  where  $\psi_0$  is an automorphism of  $\mathcal{D}$  such that  $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$  for all  $t \in T$ . The conjugation by  $\sigma$  sends the automorphism  $X \mapsto \Psi X \Psi^{-1}$  to the automorphism  $X \mapsto ({}^t\Psi^{-1})X({}^t\Psi)$ , i.e., replaces  $\Psi$  by  ${}^t\Psi^{-1}$ . Hence,  $\sigma$  commutes with  $\text{Sym}(k)$ , while the conjugation by  $\sigma$  sends  $(u_1, \dots, u_k)T$  to  $(\sigma \cdot u_1, \dots, \sigma \cdot u_k)T$ , where the action of  $\sigma$  on  $T$  is as indicated above. Also, the action of  $\sigma$  on  $G$  sends  $z_{i,j,t} := \tilde{g}_i t \tilde{g}_j^{-1}$  to  $z_{i,j,\sigma \cdot t}^{-1}$ , so  $\sigma$  belongs to  $\text{Aut}(\Gamma)$ , but not to  $\text{Stab}(\Gamma)$ . Hence we obtain  $\text{Aut}(\Gamma) = \text{Aut}(\Gamma') \rtimes \langle \sigma \rangle$  and  $\text{Stab}(\Gamma) = \text{Stab}(\Gamma')$ . The result follows.  $\square$

**Theorem 4.7.** *Let  $\mathbb{F}$  be an algebraically closed field,  $\text{char } \mathbb{F} \neq 2$ . Let  $n \geq 3$  if  $\text{char } \mathbb{F} \neq 3$  and  $n \geq 4$  if  $\text{char } \mathbb{F} = 3$ . Consider the fine grading  $\Gamma = \Gamma_A^{(\text{II})}(T, q, s, \tau)$  on  $\mathfrak{psl}_n(\mathbb{F})$  as in Definition 4.4,  $(q + 2s)\sqrt{|T|} = n$ . Let  $\Sigma = \Sigma(\tau)$ . Then  $W(\Gamma)$  contains a normal subgroup  $N$  isomorphic to  $\mathbb{Z}_2^{q+s-1}$  such that*

$$W(\Gamma)/N \cong ((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym } \Sigma \times \text{Sym}(s))) \rtimes \text{Aut}^* \Sigma,$$

where the actions are described naturally if we identify  $T^{q+s-1}$  with  $T^{q+s}/T$  and  $\mathbb{Z}_2^{q+s-1}$  with  $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$  (diagonal imbeddings). Moreover,  $W(\Gamma)$  contains a subgroup isomorphic to  $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym } \Sigma \times \text{Sym}(s))) \rtimes \text{Aut } \Sigma$  that is disjoint from  $N$ .

*Proof.* The grading  $\Gamma = \Gamma_A^{(\text{II})}(T, q, s, \tau)$  on  $\mathcal{L} = \mathfrak{psl}_n(\mathbb{F})$  is induced by the grading  $\Gamma'$  on  $\mathcal{R}^{(-)}$ , where  $\mathcal{R} = M_n(\mathbb{F})$ , obtained from  $\bar{\Gamma}' = \Gamma_{\mathcal{M}}(T, q, s, \tau)$  and  $\varphi$  as in Definition 4.4. The universal group of  $\bar{\Gamma}'$  is  $\bar{G} = \tilde{G}(T, q, s, \tau)^0$ , while the universal group of  $\Gamma$  is the extension  $G$  of  $\bar{G}$  as in Lemma 4.3. Similarly to Type I, an automorphism  $\psi'$  of  $\mathcal{R}$  sends  ${}^\alpha\Gamma'$  to  $\Gamma'$ , for some automorphism  $\alpha$  of  $G$ , if and only if the induced automorphism  $\psi$  of  $\mathcal{L}$  sends  ${}^\alpha\Gamma$  to  $\Gamma$ . Note that  $\alpha$  fixes the distinguished element  $h = (\bar{e}, -1)$  and hence yields an automorphism  $\bar{\alpha}$  of  $\bar{G}$ . It follows that  $\psi'$  sends  ${}^{\bar{\alpha}}\bar{\Gamma}'$  to  $\bar{\Gamma}'$ . For any  $g \in G$  and  $X \in \mathcal{R}_g$ , we have  $\varphi(X) = -\chi(g)X$ . Since  $(\psi')^{-1}(X) \in \mathcal{R}_{\alpha^{-1}(g)}$ , we also have  $(\varphi(\psi')^{-1})(X) = -\chi(\alpha^{-1}(g))(\psi')^{-1}(X)$ . It follows that  $\psi'\varphi(\psi')^{-1} = \xi\varphi$  where  $\xi$  is the action of the character  $(\chi \circ \alpha^{-1})\chi^{-1}$  on  $\mathcal{R}$  determined by the  $G$ -grading  $\Gamma'$ . Since  $\alpha(h) = h$ ,  $(\chi \circ \alpha^{-1})\chi^{-1}$  can be regarded as a character of  $\bar{G}$ , hence  $\xi$  belongs to  $\text{Diag}(\bar{\Gamma}')$ . Conversely, if  $\psi'$  sends  ${}^{\bar{\alpha}}\bar{\Gamma}'$  to  $\bar{\Gamma}'$  and  $\psi'\varphi(\psi')^{-1} = \xi\varphi$  for some  $\xi \in \text{Diag}(\bar{\Gamma}')$ , then for any  $\bar{g} \in \bar{G}$  and  $X \in \mathcal{R}_{\bar{g}}$ , we have  $\psi'(X) \in \mathcal{R}_{\bar{\alpha}(\bar{g})}$  and  $\varphi(\psi'(X)) = \nu\psi'(X)$  where  $\nu \in \mathbb{F}^\times$  depends only on  $\bar{g}$ . It follows that  $\psi'$  permutes the components of  $\Gamma'$  and hence sends  ${}^\alpha\Gamma'$  to  $\Gamma'$  where  $\alpha$  is a lifting of  $\bar{\alpha}$ . We have proved that an automorphism  $\psi'$  of  $\mathcal{R}$  belongs

to  $\text{Aut}^*(\bar{\Gamma}', \varphi)$ , respectively  $\text{Stab}(\bar{\Gamma}', \varphi)$ , if and only if the induced automorphism  $\psi$  of  $\mathcal{L}$  belongs to  $\text{Aut}(\Gamma)$ , respectively  $\text{Stab}(\Gamma)$ . Finally, note that  $-\varphi$  induces an automorphism of  $\mathcal{L}$  that belongs to  $\text{Stab}(\Gamma)$ . It follows that the Weyl group of  $\Gamma$  is isomorphic to  $\text{Aut}^*(\bar{\Gamma}', \varphi)/\text{Stab}(\bar{\Gamma}', \varphi)$ . The latter group was described in Theorem 3.12.  $\square$

If  $\text{char } \mathbb{F} = 3$ , there are two fine gradings on  $\mathfrak{psl}_3(\mathbb{F})$ : the Cartan grading, whose universal group is  $\mathbb{Z}^2$ , and the grading induced by the Cayley–Dickson doubling process for octonions, whose universal group is  $\mathbb{Z}_2^3$ . The Weyl groups of these gradings are, respectively, the classical Weyl group of type  $G_2$  [EKb, Theorem 3.3] and  $\text{GL}_3(2)$  [EKb, Theorem 3.5].

## 5. SERIES $B$ , $C$ AND $D$

In this section we describe the Weyl groups of fine gradings on the simple Lie algebras of series  $B$ ,  $C$  and  $D$  with exception of type  $D_4$ . Thus, we take  $\mathcal{R} = M_n(\mathbb{F})$ ,  $n \geq 4$ , and  $\mathcal{L} = \mathcal{K}(\mathcal{R}, \varphi)$  where  $\varphi$  is an involution on  $\mathcal{R}$ . If  $\varphi$  is symplectic, then, of course,  $n$  has to be even. If  $\varphi$  is orthogonal, we assume  $n \geq 5$  and  $n \neq 8$ . First we review the classification of fine gradings on  $\mathcal{L}$  from [Eld10] (extended to positive characteristic using automorphism group schemes) and then derive the Weyl groups for  $\mathcal{L}$  from what we already know about automorphisms of fine  $\varphi$ -gradings (Section 3) on  $\mathcal{R}$ .

**5.1. Classification of fine gradings.** Under the stated assumptions on  $n$ , the restriction from  $\mathcal{R}$  to  $\mathcal{L}$  yields an isomorphism  $\mathbf{Aut}(\mathcal{R}, \varphi) \rightarrow \mathbf{Aut}(\mathcal{L})$  (see [BK10, §3]). It follows that the classification of fine gradings on  $\mathcal{L}$  is the same as the classification of fine  $\varphi$ -gradings on  $\mathcal{R}$  (here  $\varphi$  is fixed).

The case of series  $B$  is quite easy, because  $n$  is odd and hence the elementary 2-group  $T$  must be trivial. Let  $G = \tilde{G}(\{e\}, q, s, \tau)^0$  where  $\tau = (e, \dots, e)$ , so  $G \cong \mathbb{Z}_2^{q-1} \times \mathbb{Z}^s$ .

**Definition 5.1.** Consider the grading  $\Gamma = \Gamma_{\mathcal{M}}(\{e\}, q, s, \tau)$  on  $\mathcal{R}$  by  $G$ . Let  $\Phi$  be the matrix given by

$$\Phi = \text{diag} \left( \underbrace{1, \dots, 1}_q, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

Then  $\Gamma$  is a fine  $\varphi$ -grading for  $\varphi(X) = \Phi^{-1}({}^tX)\Phi$  and hence its restriction is a fine grading on  $\mathcal{L} \cong \mathfrak{so}_n(\mathbb{F})$ . We will denote this grading by  $\Gamma_B(q, s)$ .

Now we turn to series  $C$  and  $D$ , where  $n$  is even and hence  $T$  may be nontrivial. So, let  $T$  be an elementary 2-group of even dimension. Choose  $\tau$  as in (1) with all  $t_i \in T_-$  in case of series  $C$  and all  $t_i \in T_+$  in case of series  $D$ . Let  $G = \tilde{G}(T, q, s, \tau)^0$ , so  $G \cong \mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0, q-1)} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s$  where  $T_0$  is the subgroup of  $T$  generated by the elements  $t_i t_{i+1}$ ,  $i = 1, \dots, q-1$ .

**Definition 5.2.** Consider the grading  $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$  on  $\mathcal{R}$  by  $G$  where  $t_1 \neq t_2$  if  $q = 2$  and  $s = 0$ . Let  $\Phi$  be the matrix given by

$$\Phi = \text{diag} \left( X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix} \right),$$

where  $\delta = -1$  for series  $C$  and  $\delta = 1$  for series  $D$ . Then  $\Gamma$  is a fine  $\varphi$ -grading for  $\varphi(X) = \Phi^{-1}({}^tX)\Phi$  and hence its restriction is a fine grading on  $\mathcal{L} \cong \mathfrak{sp}_n(\mathbb{F})$  or  $\mathfrak{so}_n(\mathbb{F})$ . We will denote this grading by  $\Gamma_C(T, q, s, \tau)$  or  $\Gamma_D(T, q, s, \tau)$ , respectively.

The following three results are Theorem 5.2 of [Eld10], stated separately for series  $B$ ,  $C$  and  $D$  (and extended to positive characteristic).

**Theorem 5.3.** *Let  $\mathbb{F}$  be an algebraically closed field,  $\text{char } \mathbb{F} \neq 2$ . Let  $n \geq 5$  be odd. Then any fine grading on  $\mathfrak{so}_n(\mathbb{F})$  is equivalent to  $\Gamma_B(q, s)$  where  $q + 2s = n$ . Also,  $\Gamma_B(q_1, s_1)$  and  $\Gamma_B(q_2, s_2)$  are equivalent if and only if  $q_1 = q_2$  and  $s_1 = s_2$ .  $\square$*

**Theorem 5.4.** *Let  $\mathbb{F}$  be an algebraically closed field,  $\text{char } \mathbb{F} \neq 2$ . Let  $n \geq 4$  be even. Then any fine grading on  $\mathfrak{sp}_n(\mathbb{F})$  is equivalent to  $\Gamma_C(T, q, s, \tau)$  where  $(q + 2s)\sqrt{|T|} = n$ . Moreover,  $\Gamma_C(T_1, q_1, s_1, \tau_1)$  and  $\Gamma_C(T_2, q_2, s_2, \tau_2)$  are equivalent if and only if  $T_1 \cong T_2$ ,  $q_1 = q_2$ ,  $s_1 = s_2$  and, identifying  $T_1 = T_2 = \mathbb{Z}_2^{2r}$ ,  $\Sigma(\tau_1)$  is conjugate to  $\Sigma(\tau_2)$  by the twisted action of  $\text{Sp}_{2r}(2)$  as in Definition 3.9.  $\square$*

**Theorem 5.5.** *Let  $\mathbb{F}$  be an algebraically closed field,  $\text{char } \mathbb{F} \neq 2$ . Let  $n \geq 6$  be even. Assume  $n \neq 8$ . Then any fine grading on  $\mathfrak{so}_n(\mathbb{F})$  is equivalent to  $\Gamma_D(T, q, s, \tau)$  where  $(q + 2s)\sqrt{|T|} = n$ . Moreover,  $\Gamma_D(T_1, q_1, s_1, \tau_1)$  and  $\Gamma_D(T_2, q_2, s_2, \tau_2)$  are equivalent if and only if  $T_1 \cong T_2$ ,  $q_1 = q_2$ ,  $s_1 = s_2$  and, identifying  $T_1 = T_2 = \mathbb{Z}_2^{2r}$ ,  $\Sigma(\tau_1)$  is conjugate to  $\Sigma(\tau_2)$  by the twisted action of  $\text{Sp}_{2r}(2)$  as in Definition 3.9.  $\square$*

**5.2. Weyl groups of fine gradings.** Let  $\Gamma = \Gamma_B(q, s)$ ,  $\Gamma_C(T, q, s, \tau)$  or  $\Gamma_D(T, q, s, \tau)$ , so  $\Gamma$  is the restriction of the grading  $\Gamma' = \Gamma_{\mathcal{M}}(T, q, s, \tau)$  on  $\mathcal{R}$  to  $\mathcal{L} = \mathcal{K}(\mathcal{R}, \varphi)$ . By arguments similar to the proof of Theorem 4.7, one shows that the Weyl group of  $\Gamma$  is isomorphic to  $\text{Aut}(\Gamma', \varphi)/\text{Stab}(\Gamma', \varphi)$ , which was described in Theorem 3.12. For  $\Gamma = \Gamma_B(q, s)$ ,  $T$  is trivial and  $\Sigma$  is a singleton of multiplicity  $q$ , so we obtain:

**Theorem 5.6.** *Let  $\mathbb{F}$  be an algebraically closed field,  $\text{char } \mathbb{F} \neq 2$ . Let  $n \geq 5$  be odd. Consider the fine grading  $\Gamma = \Gamma_B(q, s)$  on  $\mathfrak{so}_n(\mathbb{F})$  as in Definition 5.1, where  $q + 2s = n$ . Let  $\Sigma = \Sigma(\tau)$ . Then  $W(\Gamma) \cong \text{Sym}(q) \times W(s)$  where  $W(s) = \mathbb{Z}_2^s \rtimes \text{Sym}(s)$  (wreath product of  $\text{Sym}(s)$  and  $\mathbb{Z}_2$ ).  $\square$*

For  $\Gamma_C(T, q, s, \tau)$  and  $\Gamma_D(T, q, s, \tau)$ ,  $T$  may be nontrivial, so the answer is more complicated:

**Theorem 5.7.** *Let  $\mathbb{F}$  be an algebraically closed field,  $\text{char } \mathbb{F} \neq 2$ . Let  $n \geq 4$  be even. Consider the fine grading  $\Gamma = \Gamma_C(T, q, s, \tau)$  on  $\mathfrak{sp}_n(\mathbb{F})$  or  $\Gamma = \Gamma_D(T, q, s, \tau)$  on  $\mathfrak{so}_n(\mathbb{F})$  as in Definition 5.2, where  $(q + 2s)\sqrt{|T|} = n$  and  $n \neq 4, 8$  in the case of  $\mathfrak{so}_n(\mathbb{F})$ . Let  $\Sigma = \Sigma(\tau)$ . Then*

$$W(\Gamma) \cong ((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym } \Sigma \times \text{Sym}(s))) \rtimes \text{Aut } \Sigma,$$

where the actions on  $T^{q+s-1}$  are via the identification with  $T^{q+s}/T$  (diagonal imbedding).

## REFERENCES

- [BK10] Y.A. Bahturin and M. Kochetov, *Classification of group gradings on simple Lie algebras of types A, B, C and D*, J. Algebra **324** (2010), no. 11, 2971–2989.
- [Eld10] A. Elduque, *Fine gradings on simple classical Lie algebras*, J. Algebra **324** (2010), no. 12, 3532–3571.
- [EKa] A. Elduque and M. Kochetov, *Gradings on the exceptional Lie algebras  $F_4$  and  $G_2$  revisited*, to appear in Revista Matemática Iberoamericana (preprint arXiv:1009.1218 [math.RA]).

- [EKb] A. Elduque and M. Kochetov, *Weyl groups of fine gradings on matrix algebras, octonions and the Albert algebra*, preprint arXiv: 1009.1462 [math.RA].
- [HPPT02] M. Havlíček, J. Patera, E. Pelantova, and J. Tolar, *Automorphisms of the fine grading of  $\mathfrak{sl}(n, \mathbb{C})$  associated with the generalized Pauli matrices*, J. Math. Phys. **43** (2002), no. 2, 1083–1094.
- [Koc09] M. Kochetov. *Gradings on finite-dimensional simple Lie algebras*. Acta Appl. Math. **108** (2009), no. 1, 101–127.
- [PZ89] J. Patera and H. Zassenhaus, *On Lie gradings. I*, Linear Algebra Appl. **112** (1989), 87–159.

DEPARTAMENTO DE MATEMÁTICAS E INSTITUTO UNIVERSITARIO DE MATEMÁTICAS Y APLICACIONES, UNIVERSIDAD DE ZARAGOZA, 50009 ZARAGOZA, SPAIN

*E-mail address:* elduque@unizar.es

DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY OF NEWFOUNDLAND, ST. JOHN'S, NL, A1C5S7, CANADA

*E-mail address:* mikhail@mun.ca